Published by Institute of Physics Publishing for SISSA

Received: *February 20, 2006*
Accepted: *May 18, 2006*
Published: *June 6, 2006*

Relating prepotentials and quantum vacua of $\mathcal{N}=1$ gauge theories with different tree-level superpotentials

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Abstract: We consider $\mathcal{N}=1$ supersymmetric $\mathrm{U}(N)$ gauge theories with $\mathbb{Z}_k$ symmetric tree-level superpotentials $W(\Phi)$ for an adjoint chiral multiplet. We show that (for $2N/k\in\mathbb{Z}$) this $\mathbb{Z}_k$ symmetry survives in the quantum effective theory as a corresponding symmetry of the effective superpotential $W_{\rm eff}(S_i)$ under permutations of the S_i. For $W(x)=\widehat{W}(\xi)$, $\xi=x^k$, this allows us to express the prepotential $\mathcal{F}_0$ and effective superpotential $W_{\rm eff}$ on certain submanifolds of the moduli space in terms of an $\widehat{\mathcal{F}}_0$ and $\widehat{W}_{\rm eff}$ of a different theory with tree-level superpotential $\widehat{W}$. In particular, if the $\mathbb{Z}_k$ symmetric polynomial $W(x)$ is of degree $2k$, then $\widehat{W}$ is gaussian and we obtain very explicit formulae for $\mathcal{F}_0$ and $W_{\rm eff}$. Moreover, in this case, every vacuum of the effective Veneziano-Yankielowicz superpotential $\widehat{W}_{\rm eff}$ is shown to give rise to a vacuum of $W_{\rm eff}$. Somewhat surprisingly, at the level of the prepotential $\mathcal{F}_0(S_i)$ the permutation symmetry only holds for $k=2$, while it is anomalous for $k\geq 3$ due to subtleties related to the non-compact period integrals. Some of these results are also extended to general polynomial relations $\xi=h(x)$ between the tree-level superpotentials.

Keywords: Duality in Gauge Field Theories, Supersymmetric gauge theory, Matrix Models.

Contents

1. Introduction

Understanding the vacuum structure of strongly coupled gauge theories remains an important challenge. Considerable progress has been made over the last years within the framework of $\mathcal{N} = 1$ supersymmetric gauge theories in computing the exact quantum-effective superpotential W_{eff} [1]–[6]. This involved geometric engineering of the gauge theory within string theory [1, 2] and computation of the topological string amplitudes [3] on local Calabi-Yau manifolds, geometric transitions and large N dualities [4, 5], and culminated in the realisation that the non-perturbative effective superpotential W_{eff} can be directly obtained from an appropriate holomorphic matrix model in the planar limit [6]. Later on, these results were also obtained within field theory [7].

While this program has been carried out for various gauge groups and matter contents, here we will only consider the simplest case of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with $\mathrm{U}(N)$ gauge group coupled to an adjoint chiral multiplet Φ with a tree-level superpotential $W(\Phi)$. If $W(\Phi)$ is of order $n + 1$ having n non-degenerate critical points, a general vacuum breaks the gauge group to $\prod_{i=1}^{n} \mathrm{U}(N_i)$ with $\sum_{i=1}^{n} N_i = N$. This gauge theory can be obtained from IIB string theory on a specific local Calabi-Yau manifold [2] which can be taken through a geometric transition. The geometry of the local Calabi-Yau manifold after the geometric transition is directly determined by $W(x)$ together with n deformation parameters (complex structure moduli) which are encoded in the coefficients of a polynomial $f(x)$ of order $n - 1$. This geometry is closely linked to the hyperelliptic Riemann surface

$$y^2 = W'(x)^2 + f(x)\,, \tag{1.1}$$

which also appears in the planar limit of the holomorphic matrix model with action $\operatorname{tr} W(M)$. In the gauge theory, for each $\mathrm{U}(N_i)$-factor the $\mathrm{SU}(N_i)$ confines, and the low-energy dynamics is described by n $\mathrm{U}(1)$-vector multiplets together with the chiral "glueball"

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superfields $S_i \sim \mathrm{tr}_{\mathrm{SU}(N_i)} W_\alpha W^\alpha$. This is an $\mathcal{N}=2$ theory softly broken to $\mathcal{N}=1$ by some effective superpotential $W_{\mathrm{eff}}(S_i)$ which one needs to compute. Also, the $\mathrm{U}(1)^n$ couplings are given as second derivatives of a prepotential $\mathcal{F}_0(S_i)$. The effective superpotential and the prepotential are essentially given in terms of period integrals [5, 6, 8] on the Riemann surface (1.1). They can be divided into A and B periods, with the A_i periods giving (the lowest components of) the chiral superfields S_i while the B_i periods are given as $\partial\mathcal{F}_0/\partial S_i$, up to some divergent terms [8], revealing the rigid special geometry. The function $\mathcal{F}_0(S_i)$ is related to the genus zero free energy of the topological string on the local Calabi-Yau manifold and can also be identified with the matrix model planar free energy with fixed filling fractions. We will refer to it as the prepotential.

The original $\mathrm{U}(N)$ super Yang-Mills theory (in the absence of the tree-level superpotential $W(\Phi)$) has various global $\mathrm{U}(1)$ symmetries acting on Φ that are broken to discrete subgroups by the usual anomaly. However, a non-vanishing generic $W(\Phi)$ completely breaks even these remaining anomaly-free discrete symmetries. In this note, we are interested in the case where the tree-level superpotential has certain discrete symmetries and preserves a corresponding anomaly-free discrete subgroup. Our aim is to explore as much as possible the implications of these symmetries on the effective superpotential W_{eff}, as well as on the prepotential $\mathcal{F}_0$.

Suppose that $W(\Phi)$ has a $\mathbb{Z}_k$ symmetry, i.e. it is a sum of terms of the form $\mathrm{tr}\,\Phi^k$, $\mathrm{tr}\,\Phi^{2k}$, etc. Consider the $\mathrm{U}(1)$ symmetry $\Phi \to e^{i\alpha}\Phi$, with the superspace coordinate θ and the gauge multiplet W_β unaffected. (This is not an R-symmetry.) Due to the anomaly[1] this $\mathrm{U}(1)$ is broken down to $\mathbb{Z}_{2N}$, i.e. $\alpha = 2\pi\frac{r}{2N}$, $r = 1, \ldots 2N$. For a generic term in the superpotential we have $\mathrm{tr}\,\Phi^l \to e^{il\alpha}\,\mathrm{tr}\,\Phi^l = e^{2\pi i\frac{lr}{2N}}\,\mathrm{tr}\,\Phi^l$ and a general superpotential (containing at least two terms $\mathrm{tr}\,\Phi^l$ and $\mathrm{tr}\,\Phi^{l'}$ with l and l' having no common divisor) breaks the $\mathbb{Z}_{2N}$ completely. However, if $W(\Phi)$ has a $\mathbb{Z}_k$ symmetry and k divides $2N$, say $\frac{2N}{k} = s \in \mathbb{Z}$, then for all terms in $W(\Phi)$ we have $l = k\,p,\ p \in \mathbb{Z}$ and $\mathrm{tr}\,\Phi^{kp} \to e^{2\pi i p\frac{r}{s}}\,\mathrm{tr}\,\Phi^{kp}$ which is invariant for $r = s, 2s, \ldots ks = 2N$, i.e. the superpotential indeed preserves a $\mathbb{Z}_k$ subgroup of the anomaly-free $\mathbb{Z}_{2N}$.

This non-anomalous $\mathbb{Z}_k$ acts on Φ as $\Phi \to e^{2\pi i q/k}\Phi$ and, in particular, permutes among themselves the solutions of $W'(\Phi) = 0$, which are the classical vacua. Hence, it must also permute the eigenvalues sitting at (or close to) the critical points accordingly. In the effective theory which is described by the S_i one thus expects that permuting the corresponding values of the S_i is a symmetry. This is indeed the case, as we will show in this note.

In the simplest case $k = 2$, $W(x)$ is an even function of x, and then we may write $W(x) = \frac{1}{2}\widehat{W}(x^2)$. More generally, for superpotentials with $\mathbb{Z}_k$-symmetry generated by $x \to e^{2\pi i/k}x$ we write

$$W(x) = \frac{1}{k}\widehat{W}(\xi)\,, \qquad \xi = x^k\,. \tag{1.2}$$

[1]The anomalous transformation of the fermion measure gives, as usual, an extra factor $\exp(i\frac{\alpha}{8\pi^2}\times \int \mathrm{tr}_{\mathrm{ad}} F\wedge F) = \exp(i\alpha\, 2N\,\nu)$ where ν is the instanton number.

Of course, if W is of order $n+1$ and $\widehat{W}$ of order $m+1$, we must have

$$n+1 = k(m+1)\,. \tag{1.3}$$

The simplest non-trivial example is $m = 1$, $k = 2$ where $\widehat{W}$ is a quadratic (gaussian) superpotential and W a quartic one.

Our basic observation is that (1.2) induces a map between the two Riemann surfaces $\mathcal{R}$ given by $y^2 = W'(x)^2 + f(x)$ and $\widehat{\mathcal{R}}$ given by $\widehat{y}^2 = \widehat{W}'(\xi)^2 + \hat{f}(\xi)$. We will exploit this to compute and relate the corresponding period integrals, prepotentials and effective superpotentials, thus generalising the simple geometric map (1.2) to a map between two different quantum gauge theories which we will call I and II. We will systematically exploit the $\mathbb{Z}_k$ symmetry to show that there is a corresponding $\mathbb{Z}_k$ symmetry of the effective theory described by the S_i (modulo a mild anomaly discussed below). In a loose way, one might think of theory II as being the "quotient" of theory I by this $\mathbb{Z}_k$.

For general m and k, the corresponding super Yang-Mills theories have breaking patterns[2]

$$\mathrm{I} \;:\quad \mathrm{U}(N) \to \left(\prod_{l=1}^{m}\prod_{r=1}^{k}\mathrm{U}(N_{l,r})\right) \times \prod_{s=1}^{k-1}\mathrm{U}(N_{0,s}) \quad \text{and} \tag{1.4}$$

$$\mathrm{II} \;:\quad \mathrm{U}(\hat{N}) \to \prod_{j=1}^{m}\mathrm{U}(\hat{N}_j)\,. \tag{1.5}$$

Of course, in general, theory I depends on the S_i, with $i = 1, \dots n = km + k - 1$ while theory II only depends on much less fields $\hat{S}_j$, $j = 1, \dots m$. We will be able to relate the theories if precisely $k-1$ of the S_i (called $S_{0,s}$, $s = 1, \dots k-1$) vanish, and if for the remaining $S_i \equiv S_{l,r}$, with $l = 1, \dots m$ and $r = 1, \dots k$, we have $S_{l,r} = \frac{\hat{S}_l}{k}$. In particular, we will show that we can relate the prepotentials $\mathcal{F}_0$ and $\widehat{\mathcal{F}}_0$:

$$\mathcal{F}_0\left(S_{0,s} = 0;\, S_{l,r} = \frac{\hat{S}_l}{k}\right) = \frac{1}{k}\widehat{\mathcal{F}}_0(\hat{S}_l)\,. \tag{1.6}$$

Moreover, we can also relate the *effective* superpotentials $W_{\rm eff}$ and $\widehat{W}_{\rm eff}$ provided $N_{l,r} = \frac{\hat{N}_l}{k}$, $N_{0,s} = 0$ (we always take $l = 1, \dots m$ and $r = 1, \dots k$, as well as $s = 1, \dots k-1$). For these choices of N_i we will show that

$$W_{\rm eff}\left(S_{0,s} = 0;\, S_{l,r} = \frac{\hat{S}_l}{k}\right) = \frac{1}{k}\widehat{W}_{\rm eff}(\hat{S}_l)\,. \tag{1.7}$$

Note that these choices of $N_{l,r}$ and $N_{0,s}$ imply that the $\hat{N}_l$ are multiples of k and hence that N and a fortiori $2N$ is a multiple of k. This was our condition for $\mathbb{Z}_k$ to be an anomaly-free symmetry of the $\mathrm{U}(N)$ super Yang-Mills theory!

We also want to determine vacua of the quantum theory, and then one needs to find extrema of $W_{\rm eff}$ with respect to *independent* variations of all $S_{l,r}$. (The $S_{0,s}$ are not varied and remain zero if $N_{0,s} = 0$). For general $S_{l,r}$ we are able to show that the $\mathbb{Z}_k$-symmetry

[2] Note that relations between different theories having the *same* tree-level superpotential but corresponding to different gauge symmetry breaking patterns were examined e.g. in [9]. In this case $n = m$, which is quite different from the relations we are considering.

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of $W(x)$ implies a corresponding quantum symmetry[3] of $W_{\rm eff}$ under cyclic permutations $S_{l,r} \to S_{l,r+1},\ S_{l,k} \to S_{l,1}$. For the special cases of $m=1$ (and arbitrary k), this symmetry, in turn, can be exploited to show that $W_{\rm eff}$ has indeed an extremum at $S_{1,1} = \ldots S_{1,k} = S^*$ with respect to independent variations of all $S_{1,r}$, with S^* determined by the minimum of $\widehat{W}_{\rm eff}(S)$, i.e. of the Veneziano-Yankielowicz effective superpotential. One would expect that, similarly, the prepotential $\mathcal{F}_0$ is symmetric under cyclic permutations of unequal $S_{l,r}$. While this is true for $k=2$, it is no longer the case for $k \geq 3$ due to subtleties related to the common choice of cutoff for all $B_{l,r}$ cycles which breaks the $\mathbb{Z}_k$ symmetry. This is very much like an anomaly. Of course, there is nothing wrong with such an anomaly since it concerns a global discrete symmetry. Furthermore, the physical quantity in the gauge theory, $W_{\rm eff}$, is not affected by this anomaly. It would be interesting to explore whether there are physical observables beyond the gauge theory that are sensitive to this anomaly.

Note that for $m=1$ (gaussian), $\widehat{\mathcal{F}}_0$ and $\widehat{W}_{\rm eff}$ are explicitly known functions. For $m=2$, the Riemann surface is a punctured torus, and $\widehat{\mathcal{F}}_0$ and $\widehat{W}_{\rm eff}$ can still, in principle, be expressed through various combinations of complete and incomplete elliptic functions. For $m \geq 3$, *in general*, no explicit expressions in terms of special functions are known. Our mappings between theories constitute precisely the exceptions where, for $n \geq 3$ explicit expressions can nevertheless be obtained.

This paper is organised as follows. In section 2, we briefly review the formalism we will use and introduce some notation. For a detailed review we refer to [12]. We also recall some subtleties related to the definition of the non-compact (relative) B cycles and the evaluation of the corresponding period integrals (see [8] for details). In particular, we give a useful formula expressing the prepotential $\mathcal{F}_0$ solely in terms of integrals over (relative) cycles on the Riemann surface. In section 3, we establish the various relations between theories I and II. We start (section 3.1) with the simplest case of an even quartic tree-level superpotential $W(x)$ (theory I) which is mapped via $\xi = x^2$ to a gaussian tree-level superpotential $\widehat{W}(\xi)$ (theory II). This warm-up exercise already contains all the ideas but little technical complications. In particular, we relate $\mathcal{F}_0$ to $\widehat{\mathcal{F}}_0$, $W_{\rm eff}$ to $\widehat{W}_{\rm eff}$ (which is the Veneziano-Yankielowicz superpotential) for $S_1 = S_2 = \hat{S}/2 \equiv t/2$, prove the symmetries of $\mathcal{F}_0$ and $W_{\rm eff}$ under exchange of unequal S_1 and S_2, and show that each vacuum of $\widehat{W}_{\rm eff}$ (theory II) gives rise to a vacuum of $W_{\rm eff}$ (theory I). Section 3.2 deals with a general even $W(x)$ which, by $\xi = x^2$, can be mapped to a (general) $\widehat{W}(\xi)$. Here we can still relate $\mathcal{F}_0$ to $\widehat{\mathcal{F}}_0$ and $W_{\rm eff}$ to $\widehat{W}_{\rm eff}$ and prove the symmetry properties, but, in general, we do not know the *explicit* expressions of $\widehat{\mathcal{F}}_0$ or $\widehat{W}_{\rm eff}$. In secttion 3.3, we study tree-level superpotentials $W(x)$ of order $2k$ having a $\mathbb{Z}_k$-symmetry, so that one can use $\xi = x^k$ to map them to a gaussian $\widehat{W}(\xi)$. Although conceptually this is very similar to the case studied in section 3.1, there

[3] It is interesting to note that, somewhat similarly, the interplay of physical and "geometric" $\mathbb{Z}_l$ symmetries has often been useful. In $\mathcal{N}=2$ super Yang-Mills theories discrete subgroups of $\mathrm{U}(1)_R$ symmetries give rise to "geometric" $\mathbb{Z}_l$ symmetries on the moduli space of vacua which was one of the key ingredients in the determination of the spectra of stable BPS states in [10]. The arguments in the recent work [11] to relate discrete gauge invariance and the analytic structure of $\langle \det(z-\Phi) \rangle$ also uses permutations between eigenvalues located on different cuts and is somewhat reminiscent to the arguments we use in the present note.

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are various technical subtleties, related to the precise definition of the B_i cycles, which have to be addressed. In the end, we can still relate $\mathcal{F}_0$ to $\widehat{\mathcal{F}}_0$ and $W_{\rm eff}$ to $\widehat{W}_{\rm eff}$ by (1.6) and (1.7), and show, moreover, that for each vacuum of the Veneziano-Yankielowicz superpotential $\widehat{W}_{\rm eff}$ we get a vacuum of $W_{\rm eff}$. We discuss in detail how the permutation anomaly of $\mathcal{F}_0$ arises and why the symmetry is restored for $W_{\rm eff}$. Section 3.4 discusses a general $\mathbb{Z}_k$-symmetric $W(x)$ of degree $k(m+1)$. Again, relations (1.6) and (1.7) hold but, in general, we lack explicit expressions for $\widehat{\mathcal{F}}_0$ or $\widehat{W}_{\rm eff}$. Finally, in section 3.5, we comment on general maps $\xi = h(x)$ and $W(x) = \frac{1}{k}\widehat{W}(\xi)$. Equations (1.6) and (1.7) continue to be true, but without the $\mathbb{Z}_k$-symmetry we were not able to determine any vacuum of $W_{\rm eff}$ from the vacua of $\widehat{W}_{\rm eff}$, even for $m = 1$. To conclude, in section 4, we present a table summarising our results for the various cases.

2. The tools

As first conjectured in [4, 5], and motivated by the geometric transition between local Calabi-Yau manifolds [3], in general the effective superpotential $W_{\rm eff}(S_i)$ is given by

$$W_{\rm eff}(S_i) = -\sum_{i=1}^{n}\left[N_i \frac{\partial \mathcal{F}_0}{\partial S_i}(S_j) - \alpha_i(\Lambda, N_j) S_i\right] . \tag{2.1}$$

The S_i are the chiral superfields whose lowest components are the gaugino bilinears in the $\mathrm{U}(N_i)$-factors. The N_i can be interpreted, in IIB string theory, as the numbers of $D5$-branes wrapping the $i^{\rm th}$ two-cycle in the Calabi-Yau geometry before the geometric transition. After the geometric transition, the N_i are given by the integrals of the 3-form field strength $H = H_{\rm RR} + \tau H_{\rm NS}$ over the compact 3-cycles which have replaced the 2-cycles. The α_i, on the other hand, are given in terms of the integrals of the same 3-form over the non-compact 3-cycles. They can be viewed as functions of the N_i and the renormalised $\mathrm{U}(N)$ gauge-coupling constant τ, or equivalently the physical scale Λ. In particular, they are independent of the S_i which play the role of complex structure moduli. The precise form of the α_i does not concern us here. We will only need the following symmetry property: if we permute the N_j, then the α_i are permuted accordingly.[4] In particular, if all N_j are equal, then all α_i are equal, too.

The prepotential $\mathcal{F}_0$ can be obtained from the genus $(n-1)$ hyperelliptic Riemann surface given by (1.1). Here $f(x)$ is a polynomial of order $(n-1)$ depending on n coefficients in one-to-one correspondence with the S_i given by (we use S_i interchangeably to denote the superfield or its lowest component)

$$S_i = \frac{1}{4\pi i}\int_{A_i} y(x)\mathrm{d}x\,, \tag{2.2}$$

where the A_i cycle encircles clockwise the $i^{\rm th}$ cut on the upper sheet,[5] see figure 1. The prepotential $\mathcal{F}_0$ or rather $\frac{\partial \mathcal{F}_0}{\partial S_i}$ then is given in terms of integrals over *non*-compact dual

[4] This is certainly true for the Calabi-Yau geometries resulting from a $W(x)$ with a $\mathbb{Z}_k$-symmetry, as studied below. However, we are not aware of a proof of this property and we will take it as a hypothesis.

[5] Note that the way we number the cuts and corresponding cycles is different from [8]. This will simplify notations later on.

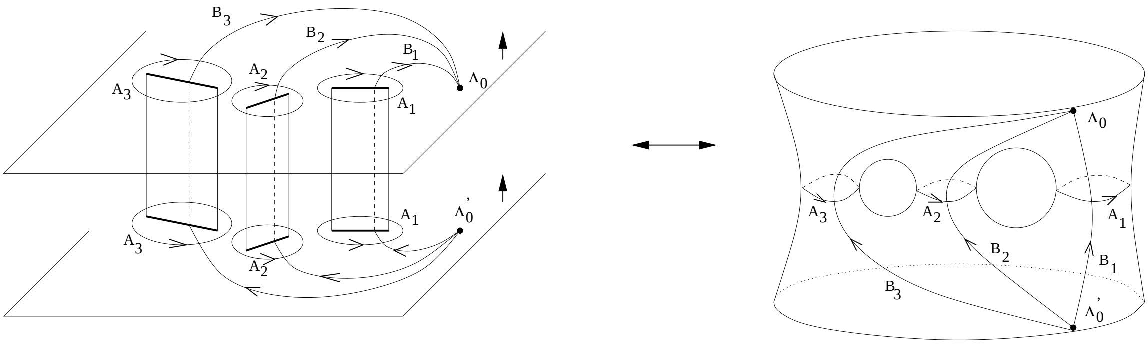

Figure 1: A symplectic choice of compact A- and non-compact B cycles for $n = 3$. Note that the orientation of the two planes on the left-hand side is chosen such that both normal vectors point to the top. This is why the orientation of the A cycles is different on the two planes. To go from the representation of the Riemann surface on the left to the one on the right one has to flip the upper plane.

cycles B_i. This involves the introduction of a cut-off Λ_0 and, as carefully discussed in [8], the *cut-off independent* result is

$$\frac{\partial \mathcal{F}_0}{\partial S_i} = \frac{1}{2}\int_{B_i} y(x)\mathrm{d}x - W(\Lambda_0) + \left(\sum_j S_j\right)\log\Lambda_0^2 + o\left(\frac{1}{\Lambda_0}\right), \tag{2.3}$$

where now B_i runs from Λ_0' on the lower sheet through the i^{th} cut to Λ_0 on the upper sheet.

The Riemann surface (1.1) also appears in the planar limit of the corresponding matrix model with *potential* $W(x)$, as has been known for a long time [13]. The physical reason why computing the effective superpotential in $\mathcal{N} = 1$ super Yang-Mills theory reduces to a matrix model, actually a holomorphic matrix model, was first discovered in [6]. The holomorphic matrix model involves integration of the eigenvalues of $\bar{N} \times \bar{N}$ matrices over a specific path $\lambda(s)$ in the complex plane[6] as discussed in detail in [14, 8]. In this context, $\mathcal{F}_0$ can be identified with the matrix model planar free energy

$$\mathcal{F}_0 = t^2\,\mathcal{P}\int \mathrm{d}s\,\mathrm{d}s'\,\log(\lambda(s)-\lambda(s'))\rho_0(s)\rho_0(s') - t\int \mathrm{d}s\;W(\lambda(s))\rho_0(s)\,, \tag{2.4}$$

where $\rho_0(s) \geq 0$ is the density of eigenvalues (with respect to the real parameter s along the path $\lambda(s)$) and it is given by

$$\rho_0(s) := \dot{\lambda}(s)\lim_{\epsilon\to 0}\frac{1}{4\pi i t}[y_+(\lambda(s)+i\epsilon\dot{\lambda}(s)) - y_+(\lambda(s)-i\epsilon\dot{\lambda}(s))]\,, \tag{2.5}$$

i.e. by the discontinuity of y_+ (y_+ denotes the value of y on the upper sheet) across its branch cuts. The parameter t is the 't Hooft coupling $t = g_s\bar{N}$ where $\frac{1}{g_s}$ is the coefficient

[6]By holomorphicity, the path $\lambda(s)$ can be chosen arbitrarily except that its asymptotics must be such that $\left|\exp\left(-\frac{1}{g_s}W(\lambda(s))\right)\right| \to 0$. However, as discussed in [8], to get a consistent saddle-point approximation (which is actually what one means by the "planar" limit), it must be such that it goes through the λ_i^* that constitute the solution of the saddle-point equations. This implies that all the cuts of y must lie on $\lambda(s)$.

in front of $W(M) = \frac{1}{n+1}\,\mathrm{tr}\, M^{n+1} + \cdots$ in the matrix model action. It is easy to check (see e.g. [8]) that this $\rho_0(s)$ is correctly normalised provided one identifies the leading coefficient of the polynomial $f(x)$ in (1.1) with $-4t$. Note also that ρ_0 depends on the S_i which are the moduli of the Riemann surface (1.1), and that $\sum_{i=1}^n S_i = t$.

In much of the matrix model literature, the parameter t is fixed to some convenient value. Here, however, it is crucial to keep t arbitrary, and hence the S_i unconstrained, so that we really have n independent moduli[7] and $\mathcal{F}_0$ is a function of all S_i. In practice, eq. (2.4) is not always convenient to actually compute $\mathcal{F}_0$. In [8], we derived an alternative formula which more directly uses the period integrals of (2.2) and (2.3), namely[8]

$$\mathcal{F}_0(S_i) = \frac{1}{2}\sum_{i=1}^n S_i \frac{\partial \mathcal{F}_0}{\partial S_i} - \frac{t}{2}\int \mathrm{d}s\, \rho_0(s) W(\lambda(s))\,. \tag{2.6}$$

The last integral reduces to a sum over integrals over the cuts. Using (2.5) it is easily rewritten as a sum of contour integrals and we get

$$\mathcal{F}_0(S_i) = \frac{1}{2}\sum_{i=1}^n \left[S_i \frac{\partial \mathcal{F}_0}{\partial S_i} - \frac{1}{4\pi i}\int_{A_i} W(x) y(x) \mathrm{d}x \right]. \tag{2.7}$$

In view of (2.2) and (2.3), this expresses $\mathcal{F}_0$ entirely in terms of integrals over the A and B cycles of the Riemann surface.

In [8] we studied how $\mathcal{F}_0$ changes under symplectic changes of basis of A and B cycles. A particularly simple symplectic change is

$$B_i \to B_i + \sum_j n_{ij} A_j\,, \quad n_{ij} = n_{ji} \in \mathbb{Z}\,. \tag{2.8}$$

It follows from (2.3) and (2.2) that $\frac{\partial \mathcal{F}_0}{\partial S_i} \to \frac{\partial \mathcal{F}_0}{\partial S_i} + 2\pi i \sum_j n_{ij} S_j$ and hence from (2.7) that [8]

$$\mathcal{F}_0 \to \mathcal{F}_0 + i\pi \sum_{i,j} S_i n_{ij} S_j\,. \tag{2.9}$$

It is quite interesting to note that equation (2.4) gives $\mathcal{F}_0$ directly in terms of the eigenvalue density ρ_0 and seems not to be concerned about how one chooses the exact form of the B_i cycles. However, it involves a double integral with a logarithm and, to be precise, one has to choose the branches of the logarithm. Choosing different branches results in adding to $\mathcal{F}_0$ a quadratic form $i\pi \sum_{i,j} S_i n_{ij} S_j$ with even integers $n_{ij} = n_{ji}$. This is in agreement with (2.9), except that only even n_{ij} appear. Indeed, the integrals in (2.4) are defined on the cut x-plane and changing the branches of the logarithm corresponds to a performing a monodromy where the cut $\mathcal{C}_i$ goes once around the cut $\mathcal{C}_j$. Under such a monodromy one has $B_i \to B_i \pm 2A_j$ and $B_j \to B_j \pm 2A_i$, necessarily with an even n_{ij}.

[7]Note that this is different from naive expectations for a compact genus $g = n-1$ Riemann surface. In particular, for $n = 1$, the sphere has no (complex structure) modulus at all. However, we are dealing with non-compact surfaces and, for $n = 1$, we actually have a sphere with marked points Λ_0 and Λ_0', or actually a sphere with two disks around the north and south pole deleted. The ratio t/Λ_0 measures the size of these holes.

[8]Eq. (3.64) of ref. [8] actually uses a different basis of cycles and is written in a slightly different but equivalent form. A similar formula also appeared in [9].

It will be useful to recall the results for the simplest case $n=1$:

$$n=1 \;\;:\quad W(x)=\frac{1}{2}(x-a)^2+w_0\,,\quad f(x)=-4t \tag{2.10}$$

Then we have a single pair of A and B cycles. From eqs. (2.2), (2.3) and (2.6) one gets

$$\begin{aligned} n=1 \;\;:\qquad S=t\,,\qquad \frac{\partial\mathcal{F}_0}{\partial t}&=t\log t-t-w_0\,,\\ \mathcal{F}_0(t)&=\frac{t^2}{2}\log t-\frac{3}{4}t^2-t\,w_0\,. \end{aligned} \tag{2.11}$$

Note that $\mathcal{F}_0$ is real for $t>0$ (and real w_0). A different choice of B cycle as in (2.8) would have resulted in a complex $\mathcal{F}_0$.

3. Relating different theories

Now we are ready to relate the free energies, resp. prepotentials, $\mathcal{F}_0$ and effective superpotentials W_{eff} of different matrix models, resp. different gauge theories.

3.1 Quartic even superpotential

As a warm-up exercise, we consider the case $n=3$ with a quartic superpotential which we require to be an even ($\mathbb{Z}_2$ symmetric) function of x:

$$W(x)=\frac{1}{4}x^4-\frac{a}{2}x^2+b\,. \tag{3.1}$$

If we let $\xi=x^2$ and $w_0=2b-\frac{a^2}{2}$ we have

$$\xi=x^2 \quad:\qquad W(x)=\frac{1}{2}\widehat{W}(\xi)\,,\quad \widehat{W}(\xi)=\frac{1}{2}(\xi-a)^2+w_0\,. \tag{3.2}$$

The quartic superpotential $W(x)$ has three critical points at the zeros of $W'(x)=x^3-ax$, i.e. $\sqrt{a}$, $-\sqrt{a}$, 0. Of course, the $\mathbb{Z}_2$ symmetry exchanges $\sqrt{a}$ and $-\sqrt{a}$ and leaves 0 invariant. Generically, this leads to three cuts[9] $\mathcal{C}_1,\mathcal{C}_2$ and $\mathcal{C}_0$ of different size, parametrised by three different S_1,S_2 and S_0 or, equivalently, by the three coefficients of $f(x)=-4tx^2+f_1x+f_0$. However, if we choose $f_1=f_2=0$, we not only respect the $\mathbb{Z}_2$ symmetry, but

$$y^2(x)=W'(x)^2+f(x)=x^2\left[(x^2-a)^2-4t\right]\equiv x^2\,\widehat{y}^2(x^2) \tag{3.3}$$

is such that the critical point at $x=0$ does not open to a branch cut, while the cuts that develop at $x=\pm\sqrt{a}$ have the same size. They both correspond to the single cut in the $\xi=x^2$-plane from $a-2\sqrt{t}$ to $a+2\sqrt{t}$. We call $\mathcal{R}$, resp. $\widehat{\mathcal{R}}$, the Riemann surfaces whose sheets are the upper and lower x-planes, resp. ξ-planes. From the preceeding construction we see that the t-moduli of both Riemann surfaces, t and $\widehat{t}$, coincide:

$$\widehat{t}=t\,. \tag{3.4}$$

[9] Again, the way we label the cuts is unimportant and only of notational convenience.

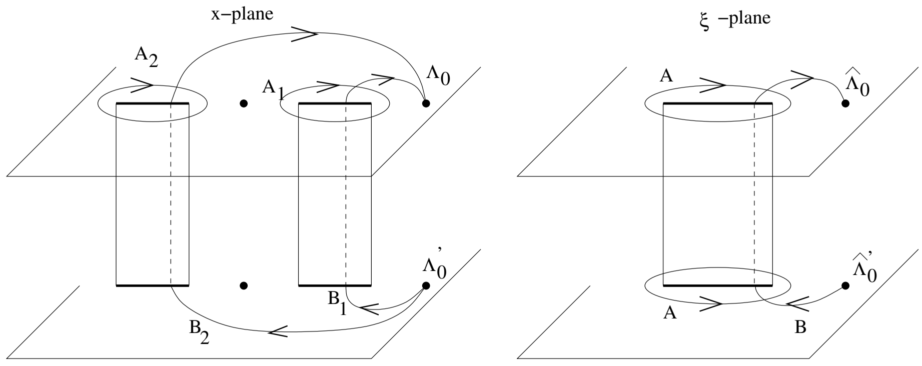

Figure 2: Shown are the cuts and cycles of $\mathcal{R}$ for the quartic superpotential (on the left) and of $\widehat{\mathcal{R}}$ for the corresponding quadratic superpotential (on the right).

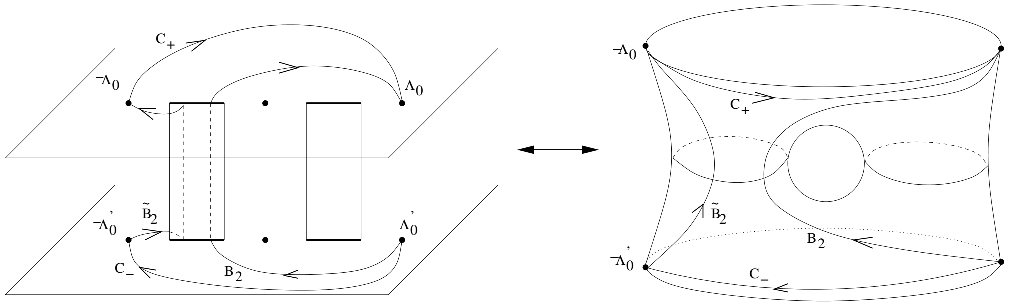

Figure 3: Shown is the decomposition of the B_2 cycle into C_-, $\widetilde{B}_2$ and C_+ for the two different representations of the Riemann surface $\mathcal{R}$.

It also follows that $y(x)$ is an *odd* function of x on both sheets and that we have

$$y(x)\,\mathrm{d}x = +\frac{1}{2}\widehat{y}(\xi)\,\mathrm{d}\xi\,. \tag{3.5}$$

As a consequence,

$$\int_{A_1} y(x)\,\mathrm{d}x = \int_{A_2} y(x)\,\mathrm{d}x = \frac{1}{2}\int_{A}\widehat{y}(\xi)\,\mathrm{d}\xi \quad \Rightarrow \quad S_1 = S_2 = \frac{t}{2}\,, \tag{3.6}$$

since both cycles A_1 and A_2 of $\mathcal{R}$ are mapped to the A cycle of $\widehat{\mathcal{R}}$, see figure 2. Obviously also, $S_0 = 0$.

For the non-compact B_1 and B_2 cycles one has to be more careful. Obviously, for the B cycle we choose start and end points $\widehat{\Lambda}_0' = (\Lambda_0')^2$ on the lower sheet and $\widehat{\Lambda}_0 = (\Lambda_0)^2$ on the upper sheet. Then the B_1 cycle is indeed mapped to the B cycle. However, this is not immediately obvious for the B_2 cycle. Instead we have

$$\int_{B_2} y(x)\,\mathrm{d}x = \int_{C_-} y(x)\,\mathrm{d}x + \int_{\widetilde{B}_2} y(x)\,\mathrm{d}x + \int_{C_+} y(x)\,\mathrm{d}x\,, \tag{3.7}$$

where C_- goes from Λ_0' to $-\Lambda_0'$ on the lower sheet, $\widetilde{B}_2$ from $-\Lambda_0'$ through the cut to $-\Lambda_0$ on the upper sheet, and C_+ from $-\Lambda_0$ to Λ_0, as indicated in figure 3.

Now, C_- is mapped to $-A$ in the ξ-plane, C_+ to A and $\widetilde{B}_2$ to B. As a result, the two integrals over C_+ and C_- cancel[10] and the integral over B_2 equals the integral over $\widetilde{B}_2$. Hence

$$\int_{B_2} y(x)\,\mathrm{d}x = \int_{B_1} y(x)\,\mathrm{d}x = \frac{1}{2}\int_B \widehat{y}(\xi)\,\mathrm{d}\xi\,. \tag{3.8}$$

Using these relations in eq. (2.3), together with $W(\Lambda_0) = \frac{1}{2}\widehat{W}(\widehat{\Lambda}_0)$ and $\log\Lambda_0^2 = \frac{1}{2}\log\widehat{\Lambda}_0^2$, as well as $\sum_i S_i = t$, yields

$$\frac{\partial\mathcal{F}_0}{\partial S_1} = \frac{\partial\mathcal{F}_0}{\partial S_2} = \frac{1}{2}\frac{\partial\widehat{\mathcal{F}}_0}{\partial t} \quad \text{at} \quad S_1 = S_2 = \frac{t}{2},\quad S_0 = 0\,. \tag{3.9}$$

Finally, we need the integrals of $W(x)y(x)\mathrm{d}x$ over the A_i cycles. By (3.2) and (3.5) they are immediately given by

$$\int_{A_i} W(x)y(x)\mathrm{d}x = \frac{1}{4}\int_A \widehat{W}(\xi)\widehat{y}(\xi)\mathrm{d}\xi\,,\quad i = 1, 2\,. \tag{3.10}$$

If we combine (3.6), (3.9) and (3.10) and use (2.7), we conclude that the planar free energy $\mathcal{F}_0$ of the matrix model with the even quartic potential (3.1) and the planar free energy $\widehat{\mathcal{F}}_0$ of the gaussian matrix model (2.10) are related as

$$\begin{aligned}\mathcal{F}_0(S_0, S_1, S_2)\Big|_{S_0=0, S_1=\frac{t}{2}, S_2=\frac{t}{2}} &= \frac{1}{2}\widehat{\mathcal{F}}_0(t)\\ &= \frac{1}{2}\left[\frac{t^2}{2}\log t - \frac{3}{4}t^2 - t\left(2b - \frac{a^2}{2}\right)\right]\,,\end{aligned} \tag{3.11}$$

where we used (2.11) and $w_0 = 2b - \frac{a^2}{2}$.

Although (3.11) is a nice result, it only gives the prepotential $\mathcal{F}_0$ on the submanifold of the moduli space where $S_1 = S_2 = \frac{t}{2}$, $S_0 = 0$. However, we now turn to the computation of the effective superpotential (2.1) and we will show that we can find vacuum configurations on this special submanifold. They are given by the points where t takes one of its vacuum values as determined by the Veneziano-Yankielowicz superpotential

$$\widehat{W}_{\mathrm{eff}}(N, t) = -N\frac{\partial\widehat{\mathcal{F}}_0}{\partial t}(t) + \widehat{\alpha}(\widehat{\Lambda}, N)\,t\,. \tag{3.12}$$

The vacua $\langle S_i\rangle$ are determined as extrema of W_{eff}, i.e.

$$\frac{\partial}{\partial S_i}W_{\mathrm{eff}}(N_i, S_i)\Big|_{S_i=\langle S_i\rangle} = 0 \quad \forall\, i = 1,\ldots n\,. \tag{3.13}$$

As to find these vacua one must compute all derivatives, one would therefore expect that the knowledge of $\mathcal{F}_0$ or W_{eff} on a particular submanifold of the moduli space is not enough. We will now show that one can nevertheless find certain vacua. To do so, it will be important to show that the $\mathbb{Z}_2$ symmetry is realised in the effective theory as the symmetry under the exchange of S_1 and S_2.

[10] It is easy to check this statement directly by taking $C_\pm$ to be large semicircles so that one can use the asymptotic form $y(x) = \pm\left(W'(x) - \frac{2t}{x}\right)$ with the $\pm$ sign on the upper/lower plane.

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First of all, note that $W_{\rm eff}$ and the vacua depend on the N_i which can be interpreted as the numbers of $D5$-branes wrapping the $i^{\rm th}$ two-cycle before the geometric transition. On the other hand, $S_i = t\bar{N}_i$ where $\bar{N}_i$ counts the number of topological branes on the $i^{\rm th}$ two-cycle. There is no direct relation between the N_i and the $\bar{N}_i$ or S_i, except when some $N_i = 0$. Then there are no $D5$-branes and hence no corresponding topological branes and no corresponding gauge group $\mathrm{U}(N_i)$, and the associated S_i must vanish. Thus if we choose $N_0 = 0$ then $S_0 = 0$ is fixed and cannot be varied.

Furthermore, we assume that N is even and make the symmetric choice $N_1 = N_2 = \frac{N}{2}$. Then, according to the remarks below eq. (2.1), we have also $\alpha_1(\Lambda; 0, \frac{N}{2}, \frac{N}{2}) = \alpha_2(\Lambda; 0, \frac{N}{2}, \frac{N}{2}) \equiv \alpha^{(3)}(\Lambda; \frac{N}{2})$ and

$$W_{\rm eff}\left(0, \frac{N}{2}, \frac{N}{2}; 0, S_1, S_2\right) = -\frac{N}{2}\left[\frac{\partial \mathcal{F}_0}{\partial S_1} + \frac{\partial \mathcal{F}_0}{\partial S_2}\right](0, S_1, S_2) + \alpha^{(3)}\left(\Lambda; \frac{N}{2}\right)[S_1 + S_2]\,. \quad (3.14)$$

In the following, we are interested in the dependence of this function on S_1 and S_2 and we often simply write $W_{\rm eff}(S_1, S_2)$. The $\mathbb{Z}_2$ symmetry of the original $\mathrm{U}(N)$ gauge symmetry acts on Φ as $\Phi \to -\Phi$ and, in particular, it must permute the eigenvalues of Φ sitting close to the two solutions at $\pm\sqrt{a}$ of $W'(\Phi) = 0$. Hence it is natural to expect that in the effective theory this $\mathbb{Z}_2$ symmetry exchanges S_1 and S_2. We now show that $W_{\rm eff}$ indeed is symmetric under interchange of S_1 and S_2. This is obviously true for the second term and has only to be shown for the first one. To get $S_2 \neq S_1$, but keep $S_0 = 0$, we must start with a more general $f(x) = -4tx^2 + f_1 x + f_0$, restricted in such a way that $y^2(x)$ still has a double zero,[11] although the latter will no longer be at $x = 0$. The general picture is still given by the left part of figure 2 but now the two cuts have different lengths and orientations. Consider also a second Riemann surface $\widetilde{\mathcal{R}}$ given by a $\widetilde{y}(x)$ with the same $W'(x)$ but with $\widetilde{f}(x) = -4tx^2 - f_1 x + f_0$ (i.e. $\widetilde{t} = t$, $\widetilde{f}_1 = -f_1$, $\widetilde{f}_0 = f_0$). Then, obviously, $\widetilde{y}^2(x) = y^2(-x)$ and, actually, $\widetilde{y}(x) = -y(-x)$ so that

$$y(x')\mathrm{d}x' = \widetilde{y}(x)\mathrm{d}x\,, \quad x' = -x\,. \quad (3.15)$$

Of course, the cuts of $\widetilde{y}$ are not the same as those of y (actually they got exchanged), but we continue to call $\mathcal{C}_1$ the cut associated with the critical point $x = \sqrt{a}$ with corresponding cycles A_1 and B_1, and to call $\mathcal{C}_2$ the cut associated with the critical point $x = -\sqrt{a}$ with corresponding cycles A_2 and B_2. So we keep the same basis of A_i and B_i cycles for both manifolds, according to figure 2. The map $x \to x' = -x$ then exchanges A_1 and A_2 as well as B_1 and $\tilde{B}_2$. It follows for the integrals over the A_i cycles that

$$\widetilde{S}_1 = \frac{1}{4\pi i}\int_{A_1} \widetilde{y}(x)\mathrm{d}x = \frac{1}{4\pi i}\int_{A_2} y(x')\mathrm{d}x' = S_2 \qquad \text{and} \qquad \tilde{S}_2 = S_1\,. \quad (3.16)$$

Since the coefficients t, f_1 and f_0 are determined by S_1, S_2 (and $S_0 = 0$) we write

$$y(x) \equiv y(x; S_1, S_2)\,, \quad \widetilde{y}(x) \equiv y(x; S_2, S_1)\,, \quad (3.17)$$

[11] For small $S_1 - S_2$ we also have small f_1 and f_0 and the double zero of y^2 only moves slightly away from $x = 0$, so that, to first order, the condition for the double zero is $f_1^2 \simeq 4(a^2 - 4t)f_0$.

($S_0 = 0$ is understood throughout), and (3.15) then reads

$$y(x'; S_1, S_2)\mathrm{d}x' = y(x; S_2, S_1)\mathrm{d}x\,, \quad x' = -x\,. \tag{3.18}$$

It then follows, with $\widetilde{B}_2$ as in figure 3,

$$\int_{\widetilde{B}_2} y(x; S_2, S_1)\mathrm{d}x = \int_{B_1} y(x'; S_1, S_2)\mathrm{d}x'\,. \tag{3.19}$$

Now it is still true that the integral over the $\widetilde{B}_2$ cycle equals the one over the B_2 cycle (since the $C_\pm$ integrals still cancel each other), and hence by (2.3) we have

$$\frac{\partial}{\partial s_2}\mathcal{F}_0(s_1, s_2)\Big|_{s_1=S_2,\ s_2=S_1} = \frac{\partial}{\partial s_1}\mathcal{F}_0(s_1, s_2)\Big|_{s_1=S_1,\ s_2=S_2}\,. \tag{3.20}$$

Similarly,[12] one has $\frac{\partial \mathcal{F}_0}{\partial S_1}(S_2, S_1) = \frac{\partial \mathcal{F}_0}{\partial S_2}(S_1, S_2)$, and it follows that $\left(\frac{\partial \mathcal{F}_0}{\partial S_1} + \frac{\partial \mathcal{F}_0}{\partial S_2}\right)(S_1, S_2)$ is symmetric under interchange of S_1 and S_2. Hence we have shown that $W_{\rm eff}$ of (3.14) is a symmetric function of S_1 and S_2. Note that this is only true because $W(x)$ is an even function of x.

Now, as for any *symmetric* function of two variables, $\frac{\partial}{\partial S} W_{\rm eff}(S, S)|_{S=S^*} = 0$ implies the vanishing of *both* partial derivatives at the symmetric point:[13]

$$\frac{\partial}{\partial S} W_{\rm eff}(S, S)\Big|_{S=S^*} = 0 \Rightarrow \frac{\partial}{\partial S_1} W_{\rm eff}(S_1, S_2)\Big|_{S_1=S_2=S^*} = \frac{\partial}{\partial S_2} W_{\rm eff}(S_1, S_2)\Big|_{S_1=S_2=S^*} = 0\,. \tag{3.21}$$

Thus, to find vacua of the gauge theory, a sufficient condition is extremality of

$$W_{\rm eff}\left(0, \frac{N}{2}, \frac{N}{2}; 0, \frac{t}{2}, \frac{t}{2}\right) = -\frac{N}{2}\frac{\partial \widehat{\mathcal{F}}_0}{\partial t}(t) + \alpha^{(3)}\left(\Lambda; \frac{N}{2}\right) t = \frac{1}{2}\widehat{W}_{\rm eff}(N, t)\,, \tag{3.22}$$

where we used the relations (3.6) and (3.9), and also identified $\alpha^{(3)}(\Lambda; \frac{N}{2}) = \frac{1}{2}\widehat{\alpha}(\widehat{\Lambda}, N)$. Note that $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}_{\rm eff}(N, t)\Big|_{t=t^*} = 0$ precisely gives the Veneziano-Yankielowicz vacua. We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}_{\rm eff}(t)\Big|_{t=t^*} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial S_1} W_{\rm eff}(S_1, S_2)\Big|_{S_1=S_2=t^*/2} = \frac{\partial}{\partial S_2} W_{\rm eff}(S_1, S_2)\Big|_{S_1=S_2=t^*/2} = 0\,. \tag{3.23}$$

Thus, we get vacuum configurations for the $\mathrm{U}(N/2) \times \mathrm{U}(N/2)$ gauge theory with a quartic tree-level superpotential from each of the Veneziano-Yankielowicz vacua. Of course, this only gives the "symmetric" vacua. We will have nothing to say for breaking patterns with $N_1 \neq N_2$ or $N_0 \neq 0$. Moreover, even for $N_1 = N_2 = N/2$, $N_0 = 0$, we expect other vacua also at $S_1 \neq S_2$.

[12] In the following we adopt the convention that $\frac{\partial \mathcal{F}_0}{\partial S_1}$ always means the derivative of $\mathcal{F}_0$ with respect to its *first argument*, etc, so that eq. (3.20) can be simply written as $\frac{\partial \mathcal{F}_0}{\partial S_2}(S_2, S_1) = \frac{\partial \mathcal{F}_0}{\partial S_1}(S_1, S_2)$.

[13] The proof is easy: suppose $g(x, y) = g(y, x)$ and $g(x, x)$ finite. We introduce $u = x + y$ and $v = x - y$. Then g is even under $v \to -v$, and $\partial_v g$ is necessarily odd and hence vanishes at $v = 0$: $\partial_v g|_{x=y} = 0$. Also $2\partial_u g|_{x=y} = [\partial_x g(x, y) + \partial_y g(x, y)]\,|_{x=y} = \frac{\mathrm{d}}{\mathrm{d}x} g(x, x)$. Hence, $\frac{\mathrm{d}}{\mathrm{d}x} g(x, x)|_{x=x^*} = 0$ implies $\partial_u g = \partial_v g = 0$ at $x = y = x^*$ and hence $\partial_x g = \partial_y g = 0$ at $x = y = x^*$.

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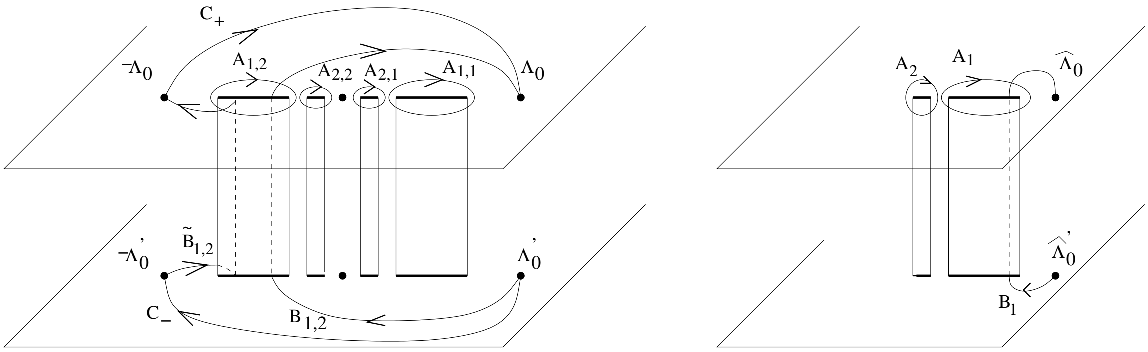

Figure 4: On the left, we have depicted the Riemann surface $\mathcal{R}$ for an even superpotential of degree 6 ($m = 2$), together with some cycles. On the right, we show the Riemann surface $\widehat{\mathcal{R}}$ for the corresponding cubic superpotential.

Finally note that not only the effective superpotential (for $N_1 = N_2, N_0 = 0$) is a symmetric function of S_1 and S_2, but the prepotential itself has the same symmetry (for $S_0 = 0$):

$$\mathcal{F}_0(0, S_2, S_1) = \mathcal{F}_0(0, S_1, S_2)\,. \tag{3.24}$$

To see this, one uses (3.18) again to show that

$$\int_{A_2} y(x; S_2, S_1) W(x) \mathrm{d}x = \int_{A_1} y(x; S_1, S_2) W(x) \mathrm{d}x \tag{3.25}$$

and hence, together with (3.20), eq. (2.7) yields (3.24).

3.2 General even superpotential

Next, we consider the case of a general even superpotential $W(x)$ of order $2m + 2$. Being even, we can always write

$$W(x) = \frac{1}{2}\widehat{W}(\xi)\,, \quad \xi = x^2\,, \tag{3.26}$$

where $\widehat{W}(\xi)$ now is of order $m + 1$. If furthermore we choose $f(x) = x^2 \widehat{f}(\xi)$ we have for $y^2(x) = W'(x)^2 + f(x)$ and $\widehat{y}^2(\xi) = \widehat{W}'(\xi)^2 + \widehat{f}(\xi)$ the same relation as before, namely $y(x)\mathrm{d}x = \frac{1}{2}\widehat{y}(\xi)\mathrm{d}\xi$. Again, we call $\mathcal{R}$ and $\widehat{\mathcal{R}}$ the Riemann surfaces corresponding to y and $\widehat{y}$, respectively. We label the cuts and cycles in such a way that the $A_{l,1}$ and $A_{l,2}$ cycles of $\mathcal{R}$ are mapped to the A_l cycle of $\widehat{\mathcal{R}}$ for all $l = 1, \ldots, m$, see figure 4. The cut that does not open is again labelled by 0. Hence

$$S_{l,1} = S_{l,2} = \frac{1}{2}\hat{S}_l\,, \quad l = 1, \ldots m\,, \quad S_0 = 0\,. \tag{3.27}$$

In particular, $t = \widehat{t}$. For the B cycles, the $B_{l,1}$ cycles of $\mathcal{R}$ are directly mapped to the B_l cycles of $\widehat{\mathcal{R}}$, while for the $B_{l,2}$ cycles things are more subtle. They have to be decomposed into cycles $\widetilde{B}_{l,2}$ which are mapped to B_l, as well as various other pieces. For example, $B_{1,2}$ is decomposed as $B_{1,2} = C_- + \widetilde{B}_{1,2} + C_+$ as shown in figure 4, with the integrals over C_- and C_+ cancelling each other and $\widetilde{B}_{1,2}$ being mapped to B_1. The $B_{2,2}$ cycle is decomposed

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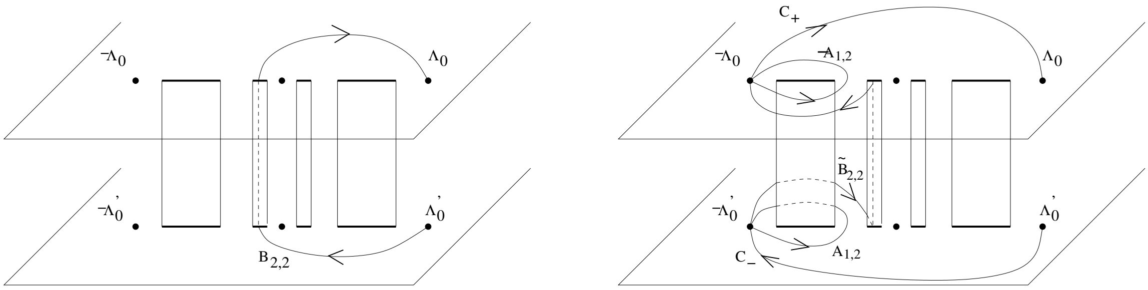

Figure 5: The decomposition of the $B_{2,2}$ cycle into $C_\pm$, $\pm A_{1,2}$ and $\tilde{B}_{2,2}$ is shown.

as follows (see figure 5). One first goes on a large arc C_- on the lower sheet to $-\Lambda_0'$ and from there one must encircle the cut $\mathcal{C}_{1,2}$ counterclockwise (which is homologous to $A_{1,2}$) before going on $\tilde{B}_{2,2}$ through the cut $\mathcal{C}_{2,2}$ to $-\Lambda_0$ on the upper sheet. There again one has to encircle the cut $\mathcal{C}_{1,2}$ counterclockwise (which on the upper sheet is homologous to $-A_{1,2}$), before going on the large arc C_+ to Λ_0. A similar decomposition applies for all $B_{l,2}$:

$$B_{l,2} = C_- + \sum_{l'=1}^{l-1} A_{l',2} + \widetilde{B}_{l,2} - \sum_{l'=1}^{l-1} A_{l',2} + C_+ \,. \tag{3.28}$$

Of course, the integrals over the $A_{l',2}$ cancel, as do those over $C_\pm$, while $\tilde{B}_{l,2}$ is mapped to B_l. Thus the $B_{l,2}$ integrals equal the $\tilde{B}_{l,2}$ integrals for all l and $\int_{B_{l,p}} y(x)\mathrm{d}x = \frac{1}{2}\int_{B_l} \widehat{y}(\xi)\mathrm{d}\xi$.

As a result, one concludes, as for the quartic superpotential, that

$$\frac{\partial \mathcal{F}_0}{\partial S_{l,1}} = \frac{\partial \mathcal{F}_0}{\partial S_{l,2}} = \frac{1}{2}\frac{\partial \widehat{\mathcal{F}}_0}{\partial \hat{S}_l} \quad \text{at} \quad S_{l,1} = S_{l,2} = \frac{1}{2}\hat{S}_l \,, \quad S_0 = 0 \,. \tag{3.29}$$

Also $\int_{A_{l,1}} W(x)y(x)\mathrm{d}x = \int_{A_{l,2}} W(x)y(x)\mathrm{d}x = \frac{1}{4}\int_{A_l} \widehat{W}(x)\widehat{y}(\xi)\mathrm{d}\xi$, and by (2.7) one has

$$\mathcal{F}_0(0, S_{l,1}, S_{l,1}) = \frac{1}{2}\widehat{\mathcal{F}}_0(\hat{S}_l) \,, \quad S_{l,1} = \frac{1}{2}\hat{S}_l \,. \tag{3.30}$$

In general, however, we do not have explicit expressions[14] for $\widehat{\mathcal{F}}_0$, contrary to the case $m = 1$.

We can exploit further the fact that $W(x)$ is an even function of x to show relations analogous to (3.18), (3.19) and (3.25). For these relations to be true it is crucial that $\int_{\tilde{B}_{l,2}} y(x)\mathrm{d}x = \int_{B_{l,2}} y(x)\mathrm{d}x$ even for $S_{l,1} \neq S_{l,2}$. From our discussion above this is obviously the case. We conclude that

$$\mathcal{F}_0(0, S_{l,1}, S_{l,2}) = \mathcal{F}_0(0, S_{l,2}, S_{l,1}) \,. \tag{3.31}$$

We can also compute the effective superpotential W_{eff} on the submanifold (3.27) of the moduli space and relate it to $\widehat{W}_{\text{eff}}$:

$$W_{\text{eff}}\left(0, \frac{N_l}{2}, \frac{N_l}{2}; 0, \frac{\hat{S}_l}{2}, \frac{\hat{S}_l}{2}\right) = \frac{1}{2}\widehat{W}_{\text{eff}}(N_l, \hat{S}_l) \,. \tag{3.32}$$

[14] As noted in the introduction, for $m = 2$, one can still, in principle, express $\mathcal{F}_0$ through various combinations of incomplete elliptic functions

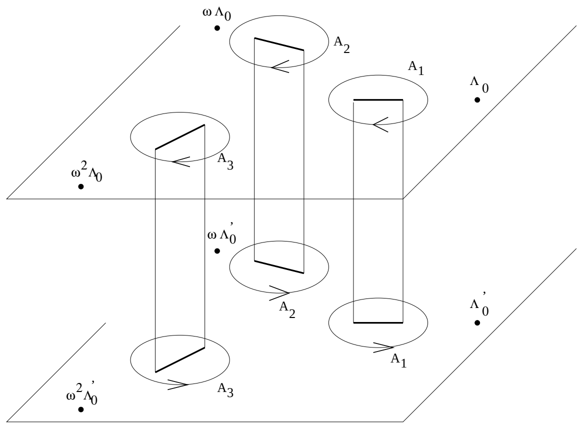

Figure 6: Shown is the Riemann surface $\mathcal{R}$ for $m = 1$, $k = 3$ with its three non-degenerate cuts $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ connecting the two sheets and the A_i cycles surrounding these cuts. We do not show the degenerate cuts corresponding to the double zeros of y^2. Note again that a clockwise oriented cycle on the upper plane is homologous to a counterclockwise oriented cycle on the lower plane. We also show the marked points Λ_0 and Λ_0' on the upper and lower sheet as well as these points rotated by $\omega = e^{2\pi i/3}$ and by $\omega^2 = e^{-2\pi i/3}$.

However, we are not able to show that the vacua of $\widehat{W}_{\rm eff}$ correspond to (some of the) vacua of $W_{\rm eff}$, although this might be expected to be true. Indeed, to prove this would require to show that the $2m$ derivatives of $W_{\rm eff}$ vanish, while extremality of $\widehat{W}_{\rm eff}$ only gives m conditions, and the symmetry of W only forces one more derivative to vanish. It is only for $m = 1$ that we have the right number of conditions.

3.3 Superpotentials of degree $2k$ with $\mathbb{Z}_k$-symmetry

Now we want to consider the general cases with $\mathbb{Z}_k$ symmetry. Start with a $W(x)$ of order $2k$, $k \geq 3$, having a $\mathbb{Z}_k$-symmetry generated by $x \to \omega x$, $\omega = e^{2\pi i/k}$. This is necessarily of the form

$$W(x) = \frac{1}{2k} x^{2k} - \frac{a}{k} x^k + b\,. \tag{3.33}$$

We let

$$\xi = x^k\,, \quad W(x) = \frac{1}{k}\widehat{W}(\xi)\,, \quad \widehat{W}(\xi) = \frac{1}{2}(\xi - a)^2 + w_0\,, \tag{3.34}$$

where $w_0 = kb - \frac{a^2}{2}$. Much of the discussion is analogous to the case of the quartic superpotential, but there are also some important differences that appear for $k \geq 3$.

We have $W'(x) = x^{k-1}(x^k - a)$ and choose $f(x) = -4tx^{2k-2}$, i.e. $f_0 = f_1 = \ldots = f_{2k-1} = 0$. Then $y^2(x) = W'(x)^2 + f(x) = x^{2(k-1)}\left[(x^k - a)^2 - 4t\right] \equiv x^{2(k-1)}\widehat{y}^2(x^k)$ and one gets k cuts $\mathcal{C}_1, \ldots \mathcal{C}_k$, as well as a multiple zero at $x = 0$. The latter corresponds to $k - 1$ degenerate cuts $\mathcal{C}_{0,1}, \ldots, \mathcal{C}_{0,k-1}$, on top of each other. The non-degenerate cuts and the corresponding $A_1, \ldots A_k$ cycles are shown in figure 6 for $k = 3$ (and $0 < 2\sqrt{t} < a$). All these $A_1, \ldots A_k$ cycles are mapped onto the single A cycle in the ξ-plane. We have

$$y(x)\mathrm{d}x = \frac{1}{k}\widehat{y}(\xi)\mathrm{d}\xi\,, \tag{3.35}$$

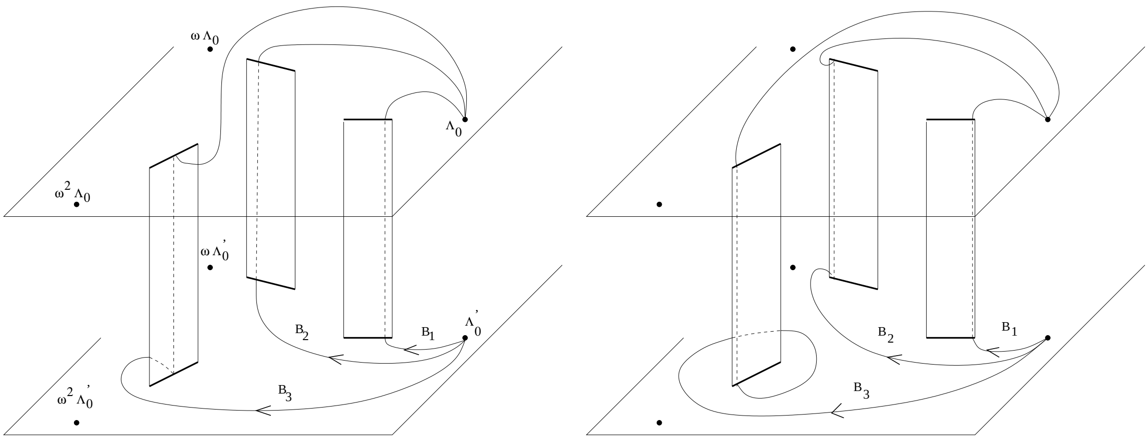

Figure 7: In the left figure we display a certain choice of B_i cycles that begin at Λ_0' on the lower sheet, go through the cut $\mathcal{C}_i$ and end at Λ_0 on the upper sheet. The right figure shows a choice of B_i cycles homologous to the one of the left figure. To see this one has to remember that a given side of a cut on the lower sheet is identified with the opposite side of the cut on the upper sheet.

and

$$\int_{A_i} y(x)\mathrm{d}x = \frac{1}{k}\int_A \widehat{y}(\xi)\mathrm{d}\xi \quad \Rightarrow \quad S_i = \frac{t}{k}\,, \quad i = 1, \dots k\,. \tag{3.36}$$

In particular, we have again $t = \sum_{i=1}^{k} S_i = \widehat{t}$. There are also $k-1$ vanishing S's corresponding to the multiple zero at $x = 0$. We will denote them

$$S_{0,1} = \dots = S_{0,k-1} = 0\,. \tag{3.37}$$

The integrals over the B_i cycles involve some subtleties, not present for $k = 2$. To see this, concentrate first on $k = 3$ and consider the choice of B_i cycles shown in figure 7 (consistent with figure 1). We want to see whether or not these cycles are mapped to the B cycle on the Riemann surface $\widehat{\mathcal{R}}$. This is obvious for B_1, but less obvious for B_2 and B_3, which must first be decomposed into various pieces. As shown in the left part of figure 8, the decomposition of B_2 is

$$B_2 \simeq C_{-,3} + A_3 + C_{-,2} + \widetilde{B}_2 + C_{+,1}\,, \tag{3.38}$$

where $C_{-,3}$ is a large arc going from Λ_0' to $\omega^2\Lambda_0'$ on the lower sheet, $C_{-,2}$ another large arc going from $\omega^2\Lambda_0'$ to $\omega\Lambda_0'$ still on the lower sheet; $\widetilde{B}_2$ goes from $\omega\Lambda_0'$ through the cut $\mathcal{C}_2$ to $\omega\Lambda_0$ on the upper sheet, and $C_{+,1}$ is a large arc on the upper sheet from $\omega\Lambda_0$ to Λ_0. Now $C_{-,3}$ and $C_{-,2}$ are both mapped to $-A$, while A_3 and $C_{+,1}$ are both mapped to A, and $\widetilde{B}_2$ is mapped to B. Hence the integrals over $C_{-,3}$, $C_{-,2}$, A_3 and $C_{+,1}$ cancel and

$$\int_{B_2} y(x)\mathrm{d}x = \int_{\widetilde{B}_2} y(x)\mathrm{d}x = \frac{1}{3}\int_B \widehat{y}(\xi)\mathrm{d}\xi\,. \tag{3.39}$$

Similarly, if we choose the B_3 cycle as shown in figure 7 it can be decomposed as in the right part of figure 8:

$$B_3 = C_{-,3} - A_3 + \widetilde{B}_3 + C_{+,2} + C_{+,1}\,, \tag{3.40}$$

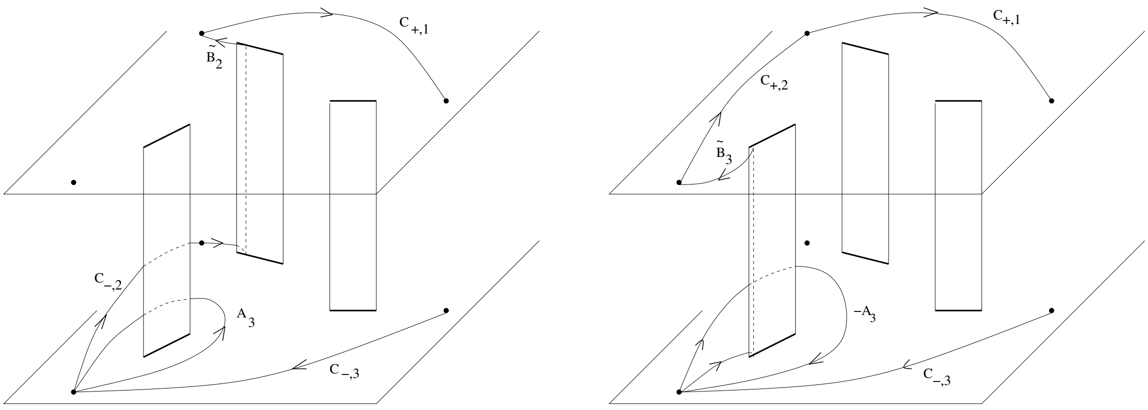

Figure 8: The left part of this figure concentrates on the B_2 cycle. It shows that it is homologous to a cycle that runs on a large arc from Λ_0' on the lower sheet to $\omega^2\Lambda_0'$, then encircles the cut $\mathcal{C}_3$ counterclockwise returning to $\omega^2\Lambda_0'$, from there goes on a large arc to $\omega\Lambda_0'$, then goes from $\omega\Lambda_0'$ through the cut $\mathcal{C}_2$ to $\omega\Lambda_0$ on the upper sheet, and from there on a large arc to Λ_0. The right part concentrates on the B_3 cycle. and shows that it is homologous to a cycle that runs on a large arc from Λ_0' on the lower sheet to $\omega^2\Lambda_0'$ ($C_{-,3}$), then encircles the cut $\mathcal{C}_3$ clockwise returning to $\omega^2\Lambda_0'$, then goes from $\omega^2\Lambda_0'$ through this same cut $\mathcal{C}_3$ to $\omega^2\Lambda_0$ on the upper sheet ($\widetilde{B}_3$), and from there on a large arc $C_{+,2}$ to $\omega\Lambda_0$, and then on $C_{+,1}$ to Λ_0.

where $\widetilde{B}_3$ goes from $\omega^2\Lambda_0'$ through the cut $\mathcal{C}_3$ to $\omega^2\Lambda_0$ and $C_{+,2}$ from $\omega^2\Lambda_0$ to $\omega\Lambda_0$. Again, $\widetilde{B}_3$ is mapped to B, while $C_{-,3}$ and $-A_3$ are mapped to $-A$, and $C_{+,2}$ and $C_{+,1}$ are mapped to A, so that the corresponding integrals cancel. The result is

$$\int_{B_3} y(x)\mathrm{d}x = \int_{\widetilde{B}_3} y(x)\mathrm{d}x = \frac{1}{3}\int_B \widehat{y}(\xi)\mathrm{d}\xi\,. \tag{3.41}$$

Note that one could have made a *choice* for B_3 different from the one shown in figure 7, e.g. not to first encircle the cut $\mathcal{C}_3$ on the lower sheet. Then one would have missed the $-A_3$ piece, resulting in an additional piece $\int_{A_3} y(x)\mathrm{d}x$ on the r.h.s. of (3.41). As discussed in section 2, such a different choice is always possible as it corresponds to the symplectic change of basis $B_3 \to B_3 + A_3$ and results in an additional piece $i\pi S_3^2$ in the prepotential. However, this extra piece spoils the "reality" of $\mathcal{F}_0$ and, more importantly, it would spoil the symmetry of $\mathcal{F}_0$ under exchange of the S_1, S_2 and S_3 to be discussed below. We conclude, that it is important for us to make precisely the choice of B_i cycles shown in figure 7.

In fact, this is easily generalised to arbitrary k. We can always consistently deform our B_i cycles into a sum of large arcs $C_{\pm,p}$ running from $\omega^p\Lambda_0$ to $\omega^{p-1}\Lambda_0$ on the lower or upper sheet, various A_j cycles and a $\tilde{B}_i$ cycle, see figure 9. More precisely, on the lower sheet we start at Λ_0' and run on a large arc $C_{-,k}$ to $\omega^{k-1}\Lambda_0'$. Then we encircle the cut $\mathcal{C}_k$ counterclockwise, which is homologous to A_k. Next, we go on another arc $C_{-,k-1}$ from $\omega^{k-1}\Lambda_0'$ to $\omega^{k-2}\Lambda_0'$, encircle the cut $\mathcal{C}_{k-1}$ counterclockwise, and so on, until we reach $\omega^{i-1}\Lambda_0'$ which is the starting point of $\tilde{B}_i$. So far there was no arbitrariness. Now we first encircle the cut $\mathcal{C}_i$ *clockwise* m_i times. This number m_i is arbitrary, a priori, but if we fix it as $m_i = i-2$ we will obtain equality of the B_i and $\tilde{B}_i$ integrals. Next, the $\tilde{B}_i$ cycle goes from

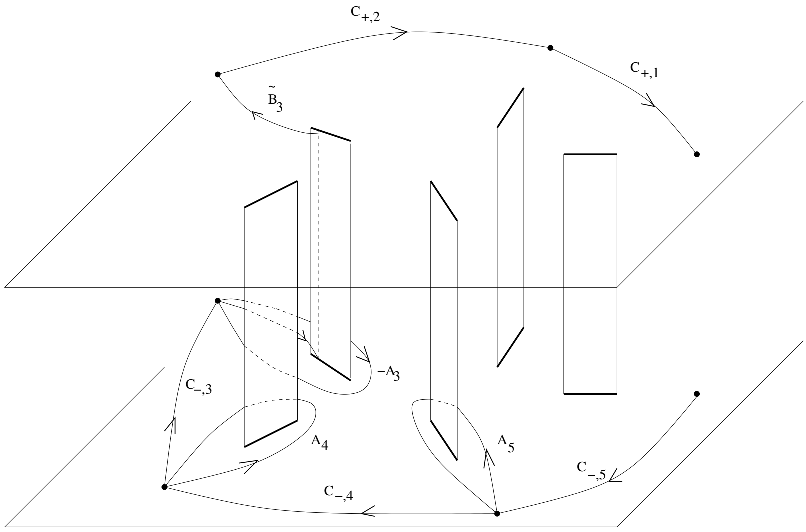

Figure 9: This figure shows, for $k = 5$ and $i = 3$ how our choice of B_i cycles is decomposed into large arcs $C_{\pm,p}$ running from $\omega^p\Lambda_0$ to $\omega^{p-1}\Lambda_0$ on the lower or upper sheet, various A_q cycles and the $\tilde{B}_i$ cycle.

$\omega^{i-1}\Lambda_0'$ through the cut $\mathcal{C}_i$ to $\omega^{i-1}\Lambda_0$ on the upper sheet. From there we go on $i-1$ large arcs $C_{+,r}$ $(r = i-1, \ldots 1)$ through $\omega^{i-2}\Lambda_0$, etc to Λ_0. The result is, for $i = 2, \ldots k$,

$$\begin{aligned} B_i &= (C_{-,k} + A_k) + (C_{-,k-1} + A_{k-1}) + \ldots + (C_{-,i+1} + A_{i+1}) \\ &\quad + (C_{-,i} - (i-2)A_i) + \tilde{B}_i + \sum_{r=1}^{i-1} C_{+,r} \,. \end{aligned} \tag{3.42}$$

Each $C_{\pm,r}$ is mapped to $\pm A$, so that $\int_{C_{\pm,r}} y(x)\mathrm{d}x = \pm \int_{A_r} y(x)\mathrm{d}x = \pm\frac{1}{k}\int_A \widehat{y}(\xi)\mathrm{d}\xi = \pm\frac{4\pi i}{k}t$ and it is immediately clear that

$$\int_{B_i} y(x)\mathrm{d}x = \int_{\widetilde{B}_i} y(x)\mathrm{d}x = \frac{1}{k}\int_B \widehat{y}(\xi)\mathrm{d}\xi \,, \tag{3.43}$$

where the B cycle runs, of course, from $\widehat{\Lambda}_0' = (\Lambda_0')^k$ to $\widehat{\Lambda}_0 = (\Lambda_0)^k$.

Using (3.43), as well as $W(\Lambda_0) = \frac{1}{k}\widehat{W}(\widehat{\Lambda}_0)$ and $\log\Lambda_0^2 = \frac{1}{k}\log\widehat{\Lambda}_0^2$, in eq. (2.3) then yields

$$\frac{\partial\mathcal{F}_0}{\partial S_1} = \frac{\partial\mathcal{F}_0}{\partial S_2} = \ldots = \frac{\partial\mathcal{F}_0}{\partial S_k} = \frac{1}{k}\frac{\partial\widehat{\mathcal{F}}_0}{\partial t} \quad \text{at } S_1 = \ldots = S_k = \frac{t}{k}\,, \quad S_{0,r} = 0\,,\; r = 1, \ldots, k-1\,. \tag{3.44}$$

Using (2.7) with (3.44), (3.36) and an analogous relation for the integrals of $W(x)y(x)\mathrm{d}x$, we finally arrive at

$$\begin{aligned} \mathcal{F}_0(S_{0,r}, S_i)\Big|_{S_{0,r}=0,\; S_i=t/k} &= \frac{1}{k}\widehat{\mathcal{F}}(t) \\ &= \frac{1}{k}\left[\frac{t^2}{2}\log t - \frac{3}{4}t - t\left(kb - \frac{a^2}{2}\right)\right] . \end{aligned} \tag{3.45}$$

Our next task is to study whether for *different* S_i the prepotential $\mathcal{F}_0$ or the effective superpotential $W_{\rm eff}$ are symmetric under cyclic permutations of the S_i. The answer will be positive for $W_{\rm eff}$ allowing us to find vacua from those of the Veneziano-Yankielowicz superpotential. However, it will be negative for $\mathcal{F}_0$ due to subtleties in the precise definition of the B_i cycles having to do with the necessity to choose a common cut-off Λ_0 for all cycles B_i. In a sense, this is like an anomaly.

We will proceed similarly to the discussion for the even quartic superpotential. Now the superpotential has a $\mathbb{Z}_k$-symmetry $W(\omega x) = W(x)$, where $\omega = e^{2i\pi/k}$. We will compare different Riemann surfaces related to each other essentially by a $\mathbb{Z}_k$ rotation $x \to \omega x$. The subtlety, however, is that Λ_0 is kept fixed, and the previously shown (non)equivalence of the $\widetilde{B}_i$ and B_i cycles will be crucial.

We start with $y^2 = W'(x)^2 + f(x)$ and

$$f(x) = -4tx^{2k-2} + \sum_{p=0}^{2k-3} f_p x^p \,, \tag{3.46}$$

where now the f_p are non-zero but still such that y^2 has $k-1$ double zeros (although they will no longer all be at $x=0$). In particular, we still have $S_{0,r} = 0\,,\ r = 1,\dots,k-1$. We let $\widetilde{f}_p = \omega^{p+2} f_p$ so that $\widetilde{t} = t$ and, with obvious notation,

$$\widetilde{f}(x) = \omega^2 f(\omega x)\,, \quad \widetilde{y}^2 = W'(x)^2 + \widetilde{f}(x)\,. \tag{3.47}$$

Then we have $\widetilde{y}^2(x) = \omega^2 y^2(\omega x)$ and

$$\widetilde{y}(x)\, \mathrm{d}x = y(x')\, \mathrm{d}x'\,, \quad x' = \omega x\,. \tag{3.48}$$

The map $x \to x' = \omega x$ maps the A_i cycle to the A_{i+1} cycle ($A_{k+1} \equiv A_1$), cf. figure 6. It follows that

$$\widetilde{S}_i = \frac{1}{4\pi i}\int_{A_i} \widetilde{y}(x)\mathrm{d}x = \frac{1}{4\pi i}\int_{A_{i+1}} y(x')\mathrm{d}x' = S_{i+1}\,. \tag{3.49}$$

Similarly, under $x \to x' = \omega x$, the $\widetilde{B}_i$ cycle is mapped to the $\widetilde{B}_{i+1}$ cycle[15] (with $\widetilde{B}_{k+1} \equiv \widetilde{B}_1 \equiv B_1$) and

$$\int_{\widetilde{B}_i} \widetilde{y}(x)\mathrm{d}x = \int_{\widetilde{B}_{i+1}} y(x')\mathrm{d}x'\,. \tag{3.50}$$

One has to be very careful here, since we have indeed shown equality of the B_i and $\widetilde{B}_i$ integrals, but only at the symmetric point where $S_1 = \ldots = S_k = \frac{t}{k}$. This is no longer true once the S_i are allowed to take different values. Then $\int_{A_j} y(x)\mathrm{d}x = 4\pi i S_j$ while $\int_{C_{\pm,r}} y(x)\mathrm{d}x = \pm\frac{4\pi i}{k}\, t$ from the asymptotics of y. It then follows from (3.42) that

$$\int_{B_i} y(x)\mathrm{d}x = \int_{\widetilde{B}_i} y(x)\mathrm{d}x + 4\pi i\left(\sum_{j=i+1}^{k} S_j - (i-2)S_i + (2i-2-k)\frac{t}{k}\right)\,, \quad i = 1,\dots k\,. \tag{3.51}$$

[15]It should be clear that the $\widetilde{B}_i$ cycles are defined as in figures 8 and 9, and, as for the discussion of the quartic superpotential, the tildes on the $\widetilde{B}_i$ have nothing to do with the tildes on $\widetilde{y}$ or $\widetilde{S}_i$. We apologize for too many tildes!

(For $i=1$ this yields correctly $\int_{B_1} y(x)\mathrm{d}x = \int_{\widetilde{B}_1} y(x)\mathrm{d}x$.) Although the individual differences are non-vanishing for each $i \neq 1$, their sum vanishes and thus

$$\sum_{i=1}^{k} \int_{B_i} y(x)\mathrm{d}x = \sum_{i=1}^{k} \int_{\widetilde{B}_i} y(x)\mathrm{d}x\,. \tag{3.52}$$

Of course, the same relation holds with y replaced by $\widetilde{y}$. Combining (3.50) and (3.52) then shows that

$$\sum_{i=1}^{k} \int_{B_i} \widetilde{y}(x)\mathrm{d}x = \sum_{i=1}^{k} \int_{\widetilde{B}_i} \widetilde{y}(x)\mathrm{d}x = \sum_{i=1}^{k} \int_{\widetilde{B}_{i+1}} y(x')\mathrm{d}x' = \sum_{j=1}^{k} \int_{\widetilde{B}_j} y(x')\mathrm{d}x' = \sum_{j=1}^{k} \int_{B_j} y(x')\mathrm{d}x'\,. \tag{3.53}$$

As discussed for the quartic superpotential, the coefficients f_p are determined by the S_i, and $\widetilde{y}(x)$ can be rewritten as $\widetilde{y}(x) \equiv y(x;\widetilde{S}_1,\widetilde{S}_2,\ldots\widetilde{S}_{k-1},\widetilde{S}_k) \equiv y(x;S_2,S_3,\ldots S_k,S_1)$ by (3.49). Then eq. (3.53) reads

$$\sum_{i=1}^{k} \int_{B_i} y(x;S_2,S_3,\ldots S_k,S_1)\mathrm{d}x = \sum_{j=1}^{k} \int_{B_j} y(x;S_1,S_2,\ldots S_{k-1},S_k)\mathrm{d}x\,, \tag{3.54}$$

and from eq. (2.3) immediately

$$\sum_{i=1}^{k} \frac{\partial}{\partial s_i}\mathcal{F}_0(s_i)\Big|_{s_r=S_{r+1}} = \sum_{j=1}^{k} \frac{\partial}{\partial s_j}\mathcal{F}_0(s_j)\Big|_{s_r=S_r}\,. \tag{3.55}$$

Note that, contrary to the case $k = 2$, we now have $\sum_{i=1}^{k} s_i \frac{\partial}{\partial s_i}\mathcal{F}_0(s_i)\Big|_{s_r=S_{r+1}}$ $\neq \sum_{i=1}^{k} s_i \frac{\partial}{\partial s_i}\mathcal{F}_0(s_i)\Big|_{s_r=S_r}$, in general, and we can no longer use (2.7) to conclude that $\mathcal{F}_0$ is symmetric under cyclic permutations of its arguments. Again, this is due to the difference of the B_i and $\widetilde{B}_i$ cycles, i.e. due to the introduction in the quantum theory of a common cutoff Λ_0 which spoils the classical $\mathbb{Z}_k$ symmetry: we have a "permutation anomaly".

Nevertheless, (3.55) is all we need in order to show the corresponding symmetry of the effective superpotential and to be able to obtain vacua. We choose $N_{0,s}=0$ (so that the $S_{0,s}$ remain zero), $N_i = \frac{N}{k}$, $i=1,\ldots k$ and denote $\alpha_i \equiv \alpha^{(2k-1)}(\Lambda;\frac{N}{k})$ so that

$$W_{\text{eff}}\left(N_{0,s}=0, N_i=\frac{N}{k}; S_{0,s}=0, S_i\right) = \sum_{i=1}^{k}\left[-\frac{N}{k}\frac{\partial\mathcal{F}_0}{\partial S_i}(S_{0,s}=0,S_i) + \alpha^{(2k-1)}\left(\Lambda;\frac{N}{k}\right)S_i\right]. \tag{3.56}$$

According to (3.55) this is invariant under cyclic permutations of the S_i. Again, due to this symmetry, W_{eff} has a critical point with respect to *independent* variations[16] of *all* S_i,

[16] Suppose that $F(s_2,\ldots s_k,s_1) = F(s_1,s_2,\ldots s_k)$. A pedestrian proof that $\frac{\mathrm{d}}{\mathrm{d}s}F(s,s,\ldots s)|_{s=s^*} = 0$ implies $\frac{\partial F}{\partial s_i}(s_1,\ldots s_k)|_{s_1=\ldots s_k=s^*} = 0$, $\forall i=1,\ldots k$, is the following: One changes variables to $u=\sum_{i=1}^k s_i$ and $v_r = \sum_{i=1}^k \omega^{ir} s_i$, $r=1,\ldots k-1$. Then under $(s_1,s_2,\ldots s_k)\to(s_2,s_3,\ldots s_1)$ one has $v_r \to \omega^{-r} v_r$ while u is invariant. Since F is invariant, it can depend arbitrarily on u, but dependence on the v_r can only be

$i = 1, \ldots k$ if (we have identified $\alpha^{(2k-1)}(\Lambda; \frac{N}{k}) = \frac{1}{k}\widehat{\alpha}(\widehat{\Lambda}, N)$)

$$\widehat{W}_{\rm eff}(N,t) = kW_{\rm eff}\left(N_{0,r} = 0, N_i = \frac{N}{k}; S_{0,r} = 0, S_i = \frac{t}{k}\right) \tag{3.57}$$

has a critical point with respect to t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}_{\rm eff}(N,t)\Big|_{t=t^*} = 0 \tag{3.58}$$

$$\Rightarrow \quad \frac{\partial}{\partial S_i}W_{\rm eff}\left(N_{0,s} = 0, N_i = \frac{N}{k}; S_{0,s} = 0, S_i\right)\Big|_{S_1=\ldots=S_k=\frac{t^*}{k}} = 0\,, \quad \forall i = 1, \ldots k\,. \tag{3.59}$$

Thus we get vacua for the $\mathrm{U}(N)$ gauge theory, broken to $\prod_{i=1}^k \mathrm{U}(\frac{N}{k})$ by a tree-level superpotential of order $2k$ having a $\mathbb{Z}_k$ symmetry, from the Veneziano-Yankielowicz vacua!

3.4 General superpotentials with $\mathbb{Z}_k$-symmetry

A general superpotential with a $\mathbb{Z}_k$-symmetry is a polynomial in $\xi = x^k$ of order $m+1$,

$$W(x) = \frac{1}{(m+1)k}x^{(m+1)k} + \sum_{r=0}^{m}\frac{g_{rk}}{rk}x^{rk}\,, \tag{3.60}$$

and it is mapped to a corresponding $\frac{1}{k}\widehat{W}(\xi)$ of order $m+1$. If we restrict to $f(x)$ of the form $f(x) = x^{2k-2}\widehat{f}(\xi)$ we have

$$y^2(x) = W'(x)^2 + f(x) = x^{2(k-1)}\left[\widehat{W}'(\xi)^2 + \widehat{f}(\xi)\right] \equiv x^{2(k-1)}\widehat{y}^2(\xi)\,. \tag{3.61}$$

Now, the Riemann surface $\widehat{\mathcal{R}}$ has m cuts with A cycles A_l, $l = 1, \ldots m$ and corresponding B cycles B_l, while the Riemann surface $\mathcal{R}$ has km (non-degenerate) cuts with A cycles $A_{l,p}$ and B cycles $B_{l,p}$ such that all $A_{l,p}$, $p = 1, \ldots k$ are mapped to A_l. For the $B_{l,p}$ cycles one must first decompose them into various large arcs, A cycles and a $\widetilde{B}_{l,p}$ cycle. This is shown in figure 10 for $k = 3$ and $m = 2$. The precise choice of the $B_{l,p}$ cycles is given by a straightforward generalisation of eqs. (3.42) and (3.28), namely for $p = 2, \ldots k$

$$\begin{aligned} B_{l,p} = &\left(C_{-,k} + \sum_{q=1}^{m}A_{q,k}\right) + \left(C_{-,k-1} + \sum_{q=1}^{m}A_{q,k-1}\right) + \ldots + \left(C_{-,p+1} + \sum_{q=1}^{m}A_{q,p+1}\right) \\ &+ \left(C_{-,p} - (p-2)\sum_{q=1}^{m}A_{q,p}\right) + \sum_{q=1}^{l-1}A_{q,p} + \tilde{B}_{l,p} - \sum_{q=1}^{l-1}A_{q,p} + \sum_{r=1}^{p-1}C_{+,r}\,. \end{aligned} \tag{3.62}$$

Then we have

$$S_{l,p} = \frac{1}{4\pi i}\int_{A_{l,p}} y(x)\mathrm{d}x = \frac{1}{4\pi i}\frac{1}{k}\int_{A_l}\widehat{y}(\xi)\mathrm{d}\xi = \frac{1}{k}\hat{S}_l\,, \tag{3.63}$$

through invariant products of the v_r. In particular, F cannot depend linearly on any of the v_r and thus $\frac{\partial}{\partial v_r}F|_{v_p=0} = 0$. But $v_p = 0\,\forall p$ is equivalent to $s_1 = s_2 = \ldots = s_k$, and we see that at the symmetric point all derivatives of F with respect to v_r automatically vanish. Furthermore $\frac{\mathrm{d}}{\mathrm{d}s}F(s,s,\ldots s) = k\frac{\partial}{\partial u}F(u,v_r)|_{v_p=0}$. Hence, vanishing of $\frac{\mathrm{d}}{\mathrm{d}s}F(s,s,\ldots s)|_{s=s^*}$ implies vanishing of all partial derivatives $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v_r}$ and hence of all $\frac{\partial F}{\partial s_i}$ at the point $s_1 = \ldots = s_k = s^*$.

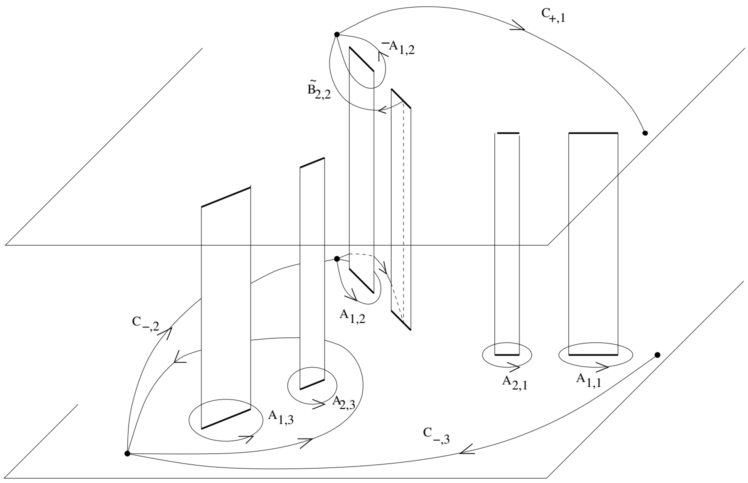

Figure 10: This figure shows, for $k = 3$ and $m = 2$, how the $B_{2,2}$ cycle is decomposed into large arcs $C_{\pm,p}$, various A cycles and the $\tilde{B}_{2,2}$ cycle. The decomposition of $B_{1,2}$ is the same except that, once at $\omega\Lambda_0'$, one goes directly on the $\tilde{B}_{1,2}$ cycle through the cut $\mathcal{C}_{1,2}$ to $\omega\Lambda_0$, without encircling the cut on the $A_{1,2}$ cycle.

and

$$\int_{B_{l,p}} y(x)\mathrm{d}x = \int_{\tilde{B}_{l,p}} y(x)\mathrm{d}x = \frac{1}{k}\int_{B_l} \widehat{y}(\xi)\mathrm{d}\xi \quad \Rightarrow \quad \frac{\partial \mathcal{F}_0}{\partial S_{l,p}} = \frac{1}{k}\frac{\partial \widehat{\mathcal{F}}_0}{\partial \hat{S}_l}\,, \tag{3.64}$$

so that by (2.7)

$$\mathcal{F}_0(S_{0,r}, S_{l,p})\Big|_{S_{0,r}=0,\ S_{l,p}=\frac{1}{k}\hat{S}_l} = \frac{1}{k}\widehat{\mathcal{F}}_0(\hat{S}_l)\,. \tag{3.65}$$

In general, however, we do not have explicit expressions for $\widehat{\mathcal{F}}_0(\hat{S}_l)$.

One can similarly relate the effective superpotentials for appropriate $N_{l,q}$, as before, and even show that W_{eff} is symmetric under simultaneous cyclic symmetries $S_{l,q} \to S_{l,q+1}$, but as for the general even superpotential, this is not enough to determine any vacua.

3.5 Superpotentials of the form $W(x) = \frac{1}{k}\widehat{W}(h(x))$

Finally, we consider the case of general mappings

$$\xi = h(x) = x^k + h_{k-2}x^{k-2} + \cdots + h_1 x\,, \qquad k \geq 3\,, \tag{3.66}$$

and superpotentials $W(x) = \frac{1}{k}\widehat{W}(\xi)$, where $\widehat{W}$ is of order $m+1$ in ξ. Note that we have set h_{k-1} and h_0 to zero by appropriate shifts of ξ and x. In particular, a quadratic map $h(x) = x^2 + h_1 x + h_0$ can always be reduced to the case studied in section 3.1. Since $W'(x) = \frac{h'(x)}{k}\widehat{W}'(\xi)$, we choose $f(x) = \left(\frac{h'(x)}{k}\right)^2 \widehat{f}(\xi)$ so that

$$y^2(x) = \left(\frac{h'(x)}{k}\right)^2 \left[\widehat{W}'(\xi)^2 + \widehat{f}(\xi)\right] \equiv \left(\frac{h'(x)}{k}\right)^2 \widehat{y}^2(\xi)\,, \tag{3.67}$$

and thus

$$y(x)\mathrm{d}x = \frac{1}{k}\widehat{y}(\xi)\mathrm{d}\xi\,, \tag{3.68}$$

as before.

Again, to each cut $\mathcal{C}_l$, $l = 1, \dots m$ of $\widehat{y}$ correspond k (non-degenerate) cuts $\mathcal{C}_{l,q}$, $q = 1, \dots k$ of y, and the cycles $A_{l,q}$ are mapped to A_l. There are also $k-1$ degenerate cuts at the zeros of $h'(x)$. The picture is still as in figure 10, but now the $\mathbb{Z}_k$-symmetry gets distorted. Obviously, eq. (3.63) continues to hold: $S_{l,p} = \frac{1}{k}\hat{S}_l$ and, in particular, $t = \widehat{t}$, while $S_{0,r} = 0$, $r = 1, \dots k-1$, corresponding to the double zeros of y^2. For the $B_{l,q}$ cycles one proceeds as follows. First, one chooses Λ_0 and defines $\widehat{\Lambda}_0 = h(\Lambda_0)$. We call $\Lambda_0^{(q)}$ the k roots of $h(x) = \widehat{\Lambda}_0$, labelled such that $\Lambda_0^{(q)} \simeq \omega^{q-1}\Lambda_0 + \mathcal{O}\left(\frac{1}{\Lambda_0}\right)$. The $\widetilde{B}_{l,q}$ cycles then go from $\Lambda_0^{(q)\prime}$ on the lower sheet through the cut $\mathcal{C}_{l,q}$ to $\Lambda_0^{(q)}$ on the upper sheet and are mapped exactly, via $\xi = h(x)$, to the B_l cycles which go from $\widehat{\Lambda}'_0$ through $\mathcal{C}_l$ to $\widehat{\Lambda}_0$. Furthermore, defining the $B_{l,q}$ cycles appropriately, they can be decomposed into various $C_{\pm,r}$, $A_{l',q'}$ and the $\widetilde{B}_{l,q}$ cycles such that $\int_{B_{l,q}} y(x)\mathrm{d}x = \int_{\widetilde{B}_{l,q}} y(x)\mathrm{d}x = \frac{1}{k}\int_{B_l}\widehat{y}(\xi)\mathrm{d}\xi$, as before. Since we still have $W(\Lambda_0) = \frac{1}{k}\widehat{W}(\widehat{\Lambda}_0)$ and $\log\Lambda_0^2 = \frac{1}{k}\log\widehat{\Lambda}_0^2 + \mathcal{O}\left(\frac{1}{\Lambda_0^2}\right)$ we conclude again that $\frac{\partial\mathcal{F}_0}{\partial S_{l,q}} = \frac{1}{k}\frac{\partial\widehat{\mathcal{F}}_0}{\partial\hat{S}_l}$ at $S_{l,q} = \frac{1}{k}\hat{S}_l$, $S_{0,s} = 0$ and, hence $\mathcal{F}_0(S_{0,s} = 0;\, S_{l,q} = \frac{1}{k}\hat{S}_l) = \frac{1}{k}\widehat{\mathcal{F}}_0(\hat{S}_l)$, as before. Also, the relation between the effective superpotentials continues to hold, provided one makes the symmetric choice of the $N_{l,q}$.

If we specialise to the case $m = 1$ where $\widehat{W}$ is a gaussian superpotential, i.e. for a

$$W(x) = \frac{1}{2k}h(x)^2 - \frac{a}{k}h(x) + b\,, \tag{3.69}$$

we know, of course, the exact expression of $\widehat{\mathcal{F}}_0(t)$. In this case, one might ask further whether one can still prove some permutation symmetry of W_{eff}, for unequal S_i, and use this to find vacua. However, above, we exploited the $\mathbb{Z}_k$-symmetry of $W(x)$ to prove the symmetry under circular permutations of the S_i, and it seems unlikely that one can proceed without it.

4. Conclusions

In this note we studied relations between effective superpotentials (as well as prepotentials) of $\mathcal{N} = 1$ U(N) gauge theories with different tree-level superpotentials W and $\widehat{W}$ for an adjoint chiral multiplet, with particular emphasis on W's that preserve an anomaly-free $\mathbb{Z}_k$ symmetry $\Phi \to e^{2\pi i/k}\Phi$. These tree-level superpotentials which are polynomials of order $k(m+1)$ and $m+1$, respectively, are related by $W(x) = \widehat{W}(\xi(x))$. The determination of the effective superpotentials is essentially reduced to the computation of various period integrals on corresponding Riemann surfaces $\mathcal{R}$ and $\widehat{\mathcal{R}}$, and $\xi(x)$ constitutes a map between them. For a "general" degree k polynomial $\xi(x)$, this mapping provides the relation between the effective superpotentials of the gauge theories, but also between the prepotentials or the free energies $\mathcal{F}_0$ and $\widehat{\mathcal{F}}_0$ of the corresponding holomorphic matrix models in the planar limit. On the "symmetric" submanifold of moduli space given by $S_{l,r} = \frac{1}{k}\hat{S}_l$ and $S_{0,s} = 0$ we could express $\mathcal{F}_0$ and W_{eff} entirely in terms of $\widehat{\mathcal{F}}_0$ and $\widehat{W}_{\text{eff}}$. Moreover, in the $\mathbb{Z}_k$ symmetric case $\xi = x^k$, for unequal $S_{l,r}$, we could prove, for $k = 2$, symmetry of $\mathcal{F}_0$ under exchange of the arguments $S_{l,1} \leftrightarrow S_{l,2}$ which is the quantum manifestation of the $\mathbb{Z}_2$ symmetry in this case. For $k \geq 3$ the $\mathbb{Z}_k$ symmetry does not completely survive at the quantum level and the

$l = 1, \dots m$ $r = 1, \dots k,\;\; s = 1, \dots k-1$	Sect. 3.1 $m = 1$ $k = 2$	Sect. 3.2 $m \geq 2$ $k = 2$	Sect. 3.3 $m = 1$ $k \geq 3$	Sect. 3.4 $m \geq 2$ $k \geq 3$	Sect. 3.5 $m \geq 1$ $k \geq 3$
map	$\xi = x^2$	$\xi = x^2$	$\xi = x^k$	$\xi = x^k$	$\xi = h(x)$
$\mathcal{F}_0(S_{0,s} = 0;\, S_{l,r} = \frac{\hat{S}_l}{k}) = \frac{1}{k}\widehat{\mathcal{F}}_0(\hat{S}_l)$	yes	yes	yes	yes	yes
$W_{\text{eff}}(S_{0,s} = 0;\, S_{l,r} = \frac{\hat{S}_l}{k})$ $= \frac{1}{k}\widehat{W}_{\text{eff}}(N_l; \hat{S}_l)$ at $N_{l,r} = \frac{N_l}{k},\;\; N_{0,s} = 0$	yes	yes	yes	yes	yes
$\mathcal{F}_0(S_{0,s} = 0;\, S_{l,1}, S_{l,2}, \dots S_{l,k})$ $= \mathcal{F}_0(S_{0,s} = 0;\, S_{l,k}, S_{l,1}, \dots S_{l,k-1})$	yes	yes	no	no	no
$W_{\text{eff}}(S_{0,s} = 0;\, S_{l,1}, S_{l,2}, \dots S_{l,k})$ $= W_{\text{eff}}(S_{0,s} = 0;\, S_{l,k}, S_{l,1}, \dots S_{l,k-1})$ at $N_{l,r} = \frac{N_l}{k},\;\; N_{0,s} = 0$	yes	yes	yes	yes	no
all vacua of $\widehat{W}_{\text{eff}}$ yield vacua of W_{eff}	yes	?	yes	?	?

Table 1: The table summarises our results for the different pairs of superpotentials.

corresponding permutation symmetry $S_{l,r} \to S_{l,r+1}$ of $\mathcal{F}_0$ is anomalous due to subtleties in the precise definition of the non-compact period integrals and the necessity to introduce a common "cut-off" Λ_0 for all of them. However, the anomalous term is irrelevant when looking only at the effective superpotential for symmetric gauge group breaking patterns, i.e. $N_{l,r} = \frac{1}{k}N_l$ and $N_{0,s} = 0$, and the permutation symmetry is restored for all k. This in turn allowed us to show, for $m = 1$, that for each vacuum of $\widehat{W}_{\text{eff}}$ (which in this case is the Veneziano-Yankielowicz superpotential) there is a corresponding vacuum of W_{eff}. All this is summarised in table 1.

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